

Brief solutions to selected problems in homework 07

1. Section 14.3: Solutions, common mistakes and corrections:

(a) $1^\circ \frac{\partial f}{\partial y}(x,0) = \lim_{y \rightarrow 0} \frac{f(x,y) - f(x,0)}{y} = \lim_{y \rightarrow 0} x \cdot \frac{x^2 - y^2}{x^2 + y^2} = x \neq$
 $2^\circ \frac{\partial f}{\partial x}(0,y) = \lim_{x \rightarrow 0} \frac{f(x,y) - f(0,y)}{x} = \lim_{x \rightarrow 0} y \cdot \frac{x^2 - y^2}{x^2 + y^2} = -y \neq$

(b) $1^\circ \frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{x \rightarrow 0} \frac{f_y(x,0) - f_y(0,0)}{x} = 1$
 $2^\circ \frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{y \rightarrow 0} \frac{f_x(0,y) - f_x(0,0)}{y} = -1$

$\therefore \frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0) \neq$ ✓

Figure 1: Solution to Section 14.3, problem 72

$f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x-0} = \frac{0-0}{x-0} = 0$
 $f_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y-0} = \frac{0-0}{y-0} = 0$

try $x = ky^2$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x) = \lim_{(x,y) \rightarrow (0,0)} \frac{ky^4}{k^2y^4 + y^4} = \frac{k}{k^2 + 1}$

$\Rightarrow f$ not conti. at $(0,0)$ depend on k
 \Rightarrow not diff. at $(0,0)$ \Rightarrow doesn't exist.

Figure 2: Solution to Section 14.3, problem 91

2. Let $\Delta x = x - x_0, \Delta y = y - y_0$

(\Rightarrow). Assume $h(x,y) = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y = \left(\frac{\varepsilon_1 \Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} + \frac{\varepsilon_2 \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \right) \cdot \sqrt{\Delta x^2 + \Delta y^2}$

Since $-1 \leq \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} \cdot \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \leq 1$ (*)

$\Rightarrow -(\varepsilon_1 + \varepsilon_2) \leq \varepsilon \leq \varepsilon_1 + \varepsilon_2 \xrightarrow[\text{Thm}]{\text{Sandwich's Thm}} \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \varepsilon = 0 \quad \#$

(\Leftarrow). Assume $h(x,y) = \varepsilon \cdot \sqrt{\Delta x^2 + \Delta y^2} = \varepsilon \left(\frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} \cdot \Delta x + \frac{\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \cdot \Delta y \right)$

Let $\varepsilon_1 = \frac{\varepsilon \Delta x}{\sqrt{\Delta x^2 + \Delta y^2}}, \varepsilon_2 = \frac{\varepsilon \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \Rightarrow h(x,y) = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

Moreover by (*), $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} (\varepsilon_1, \varepsilon_2) = (0,0)$ by Sandwich's thm

Figure 3: Solution to homework 7, problem 2

2. Section 14.4: Solutions, common mistakes and corrections:

$w = \ln(x^2 + y^2 + z^2), x = ue^v \sin u, y = ue^v \cos u, (u,v) = (-2,0), z = ue^v$

$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{dx}{du} + \frac{\partial w}{\partial y} \frac{dy}{du} + \frac{\partial w}{\partial z} \frac{dz}{du} = (w_x, w_y, w_z) \cdot \left(\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du} \right)$

$= \left(\frac{2x}{x^2 + y^2 + z^2} \right) \cdot e^v (u \cos u + \sin u) + \left(\frac{2y}{x^2 + y^2 + z^2} \right) \cdot e^v (r \cos u - u \sin u) + \left(\frac{2z}{x^2 + y^2 + z^2} \right) \cdot e^v$

$= \frac{\sin 2}{2} \cdot 1 \cdot (-2 \cos 2 - \sin 2) - \frac{\cos 2}{2} \cdot 1 \cdot (\cos 2 - 2 \sin 2) - \frac{1}{2}$

$= -\left(\frac{1}{2}\right) - \frac{1}{2} = -1$

$\frac{\partial w}{\partial v} = \frac{\sin 2}{2} (u \sin e^v) - \frac{\cos 2}{2} (u \cos e^v) - \frac{1}{2} u$

$= \sin^2 2 + \cos^2 2 - \frac{1}{2} \cdot 1 \cdot 2 = 1 + 1 - 2 = 0$

Figure 4: Solution to Section 14.4, problem 10

Method 2

$w = \ln(x^2 + y^2 + z^2) = \ln(2u^2 e^{2v})$

$= \ln 2 + 2 \ln u + 2v$

$\frac{\partial w}{\partial u} = \frac{2}{u}, \frac{\partial w}{\partial v} = 2$

Figure 5: Solution to Section 14.4, problem 10, method 2

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0 \quad \checkmark$$

Figure 6: Solution to Section 14.4, problem 43

⑤

$$\text{Fact } F(x) = \int_a^b g(t, x) dt = g(u, x) \cdot \frac{du}{dx} + \int_a^u g_x(t, x) dt$$

$$\Rightarrow F'(x) = \int_a^b g_x(t, x) dt$$

⑥

$$F(x) = \int_0^{x^2} \sqrt{t^4 + x^3} dt$$

$$\Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3} \cdot \frac{d(x^2)}{dx} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}} dt$$

For $F(x) = \int_a^{f(x)} g(t, x) dt$

$$\Rightarrow G(u, x) = \int_a^u g(t, x) dt$$

$$\Rightarrow \frac{d(G(u, x))}{dx} = \underline{G_u} \cdot \frac{du}{dx} + \underline{G_x} \cdot \frac{dx}{dx}$$

Figure 7: Solution to Section 14.4, problem 51

3. Section 14.5: Solutions, common mistakes and corrections:

$$\begin{aligned}
 4: D_u f(1,1) &= (3, -4) \cdot (a, b) = 4, \quad 3a - 4b = 4 \quad \text{--- (1)} \\
 -3: D_u f(1,1) &= (3, -4) \cdot (c, d) = -3, \quad 3c - 4d = -3 \\
 \begin{cases} \textcircled{1} \\ a^2 + b^2 = 1 \end{cases} &\Rightarrow b = \frac{-1}{25} \text{ or } -1 \Rightarrow u = \left\langle \frac{24}{25}, \frac{-1}{25} \right\rangle \text{ or } \langle 0, -1 \rangle \\
 \begin{cases} \textcircled{2} \\ c^2 + d^2 = 1 \end{cases} &\Rightarrow u = \left\langle \frac{7}{25}, \frac{24}{25} \right\rangle \text{ or } \langle -1, 0 \rangle
 \end{aligned}$$

Figure 8: Solution to Section 14.5, problem 29 (d,e)

$$\begin{aligned}
 D_u f(1,2) &= 2\sqrt{2} \\
 \rightarrow \nabla f &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = 2\sqrt{2} \\
 \rightarrow \nabla f \cdot (0, -1) &= -3 \\
 \begin{cases} \frac{1}{\sqrt{2}} f_x + \frac{1}{\sqrt{2}} f_y = 2\sqrt{2} \\ 0 \cdot f_x - f_y = -3 \end{cases} \\
 \Rightarrow f_y + 3 &= 4 \\
 f_x = 1, f_y &= 3 \quad \nabla f = (1, 3) \\
 (1, 3) \cdot \frac{1}{\sqrt{5}} (-1, -2) &= \frac{-7}{\sqrt{5}} \#
 \end{aligned}$$

Figure 9: Solution to Section 14.5, problem 35

$$\begin{aligned}
 (a) \\
 |\nabla f| &= 2\sqrt{3} \\
 v &= (1, 1, -1) \\
 |v| &= \sqrt{3} \\
 \hat{v} &= \frac{v}{|v|} \\
 \nabla f &= 2\sqrt{3} \cdot \frac{1}{\sqrt{3}} (1, 1, -1) \\
 &= (2, 2, -2) \quad \checkmark \\
 (b) \\
 (+) \\
 u &= (1, 1, 0) \\
 \hat{u} &= \frac{u}{\sqrt{2}} \\
 D_u f &= (2, 2, -2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\
 &= 2\sqrt{2} \quad \checkmark
 \end{aligned}$$

Figure 10: Solution to Section 14.5, problem 36