

Brief solutions to selected problems in homework 06

1. Section 14.2: Solutions, common mistakes and corrections:

43.

$$f(x,y) = \frac{x^4 - y^2}{x^4 + y^2}$$
$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^2}{x^4 + y^2}$$

$y = kx^2$

$$= \lim_{x \rightarrow 0} \frac{x^4 - k^2 x^4}{x^4 + k^2 x^4}$$
$$= \frac{1 - k^2}{1 + k^2}$$

limit depends on $k \rightarrow$ not exist

Figure 1: Solution to Section 14.2, problem 43

48.

$$h(x,y) = \frac{x^2 y}{x^4 + y^2}$$

let $(x,y) \rightarrow (0,0)$ along
 $y = mx^2, m \in \mathbb{R}$

$$\lim_{(x,y) \rightarrow (0,0)} h(x) = \frac{mx^4}{x^4 + m^2 x^4} = \frac{m}{m^2 + 1}$$

we see $m \in \mathbb{R}$ gives diff. limit,
by Two Path Theorem, $\lim_{(x,y) \rightarrow (0,0)} \text{DNE}$

Figure 2: Solution to Section 14.2, problem 48

Approach along the line: $(y-1) = m(x-1)$
 $\Rightarrow y = m(x-1) + 1$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{xy^2 - 1}{y-1} = \lim_{x \rightarrow 1} \frac{x(m^2(x-1)^2 + 2m(x-1) + 1) - 1}{m(x-1)}$$

$$= \lim_{x \rightarrow 1} \left(\frac{xm^2(x-1)^2}{m(x-1)} + \frac{2mx(x-1)}{m(x-1)} + \frac{(x-1)}{m(x-1)} \right)$$

$$= 2 + \frac{1}{m}$$

Figure 3: Solution to Section 14.2, problem 49

$f(x,y) = \begin{cases} 1, & y \geq x^2 \\ 0, & y < 0 \end{cases}$
 $f(x) = \begin{cases} 1, & y \geq 0 \\ 0, & y < 0 \end{cases}$

(a) $(0,1)$ $\left(\begin{array}{l} \because f(x,y) = 1 \\ \text{when } (x,y) \text{ near} \\ (0,1) \end{array} \right)$
 $y \geq x^2$
limit = 1

(b) $(2,3)$
 $y < x^2$ and $y > 0$
limit = 0 $\rightarrow \left(\begin{array}{l} \because f(x,y) = 0 \text{ when} \\ (x,y) \text{ near } (2,3) \end{array} \right)$

(c) $(0,0)$
 $y = 0 \rightarrow 1$
 $y = \frac{x^2}{2} \rightarrow 0$
 $0 < \frac{x^2}{2} < x^2$

doesn't exist

Figure 4: Solution to Section 14.2, problem 51

by sandwich Thm

$$\Rightarrow \begin{cases} y \leq y \sin(\frac{1}{x}) \leq -y & y < 0 \\ -y \leq y \sin(\frac{1}{x}) \leq y & y > 0 \end{cases}$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} y = \lim_{(x,y) \rightarrow (0,0)} (-y) = 0$$

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} y \sin(\frac{1}{x}) = 0 \quad \checkmark \#$$

Figure 5: Solution to Section 14.2, problem 57

$x = r \cos \theta$, $y = r \sin \theta$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \frac{\cos^3 \theta - \cos \theta \sin^2 \theta}{\cos^2 \theta}}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$

$$= \lim_{r \rightarrow 0} \frac{r (\cos^3 \theta - \sin^2 \theta \cos \theta)}{\cos^2 \theta + \sin^2 \theta} = \lim_{r \rightarrow 0} r (\cos^3 \theta - \sin^2 \theta \cos \theta)$$

$\therefore |\cos^3 \theta + \cos^2 \theta| \leq 1$ \therefore by Sandwich's Thm.

Figure 6: Solution to Section 14.2, problem 61

$$x = mx$$

$$\lim_{x \rightarrow 0} f = \lim_{x \rightarrow 0} \frac{m^2 x^2}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{m^2}{1 + m^2}$$

$\therefore m \notin \mathbb{R} \rightarrow \lim_{x \rightarrow 0} \frac{m^2}{1 + m^2} \notin \mathbb{R}$

\therefore by two path theorem
 \rightarrow no limit ✓

(Sol-2) Polar coord.

$$\lim_{r \rightarrow 0^+} \frac{r^2 \sin^2 \theta}{r^2} = \sin^2 \theta \Rightarrow \text{DNE}$$

Figure 7: Solution to Section 14.2, problem 63

$$\lim_{(x,y) \rightarrow (0,0)} \ln \left(\frac{3x^2 - xy^2 + 3y^2}{x^2 + y^2} \right)$$

$$= \lim_{r \rightarrow 0} \ln \left(3 + \frac{r^4 \sin^2 \theta \cos^2 \theta}{r^2} \right) \stackrel{(\ominus)}{=} \ln 3$$

$(\hat{=}) \quad x = r \cos \theta, \quad y = r \sin \theta$

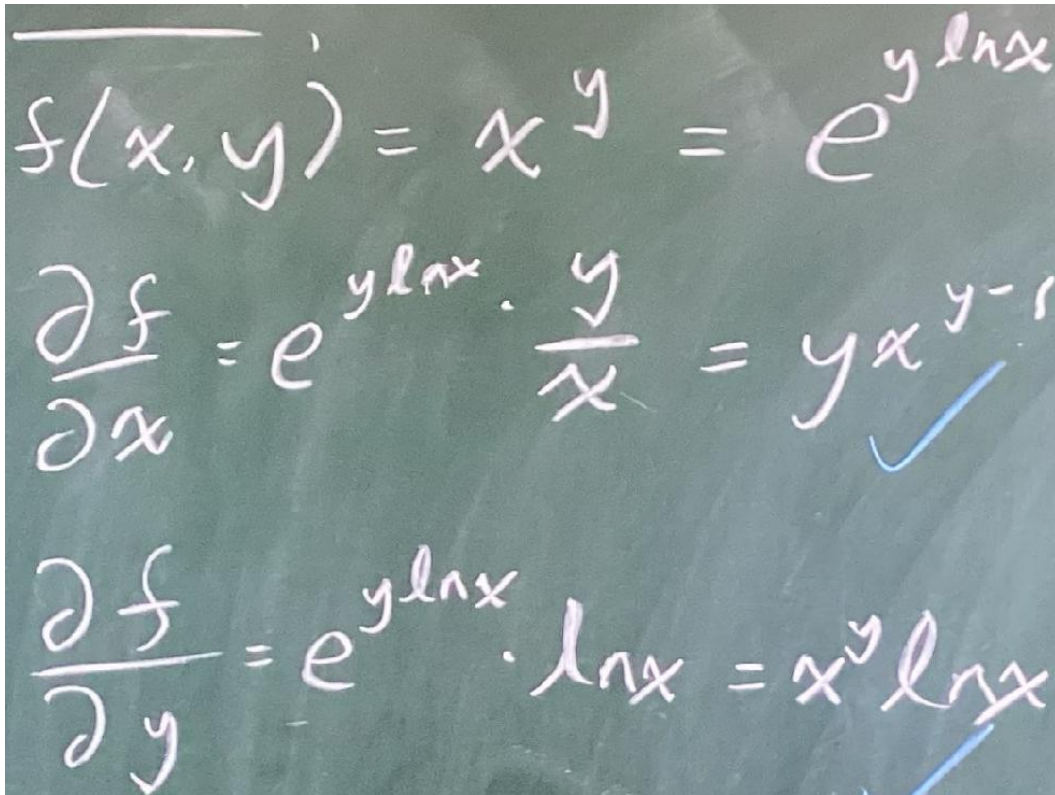
$$\left(\frac{r^4 \sin^2 \theta \cos^2 \theta}{r^2} = r^2 \sin^2 \theta \cos^2 \theta, \quad |\sin \theta \cos \theta| \leq 1 \right)$$

\therefore Sandwich's Thm $\Rightarrow \lim_{r \rightarrow 0^+} r^2 = 0$

\therefore Define $f(0,0) = \ln 3$

Figure 8: Solution to Section 14.2, problem 67

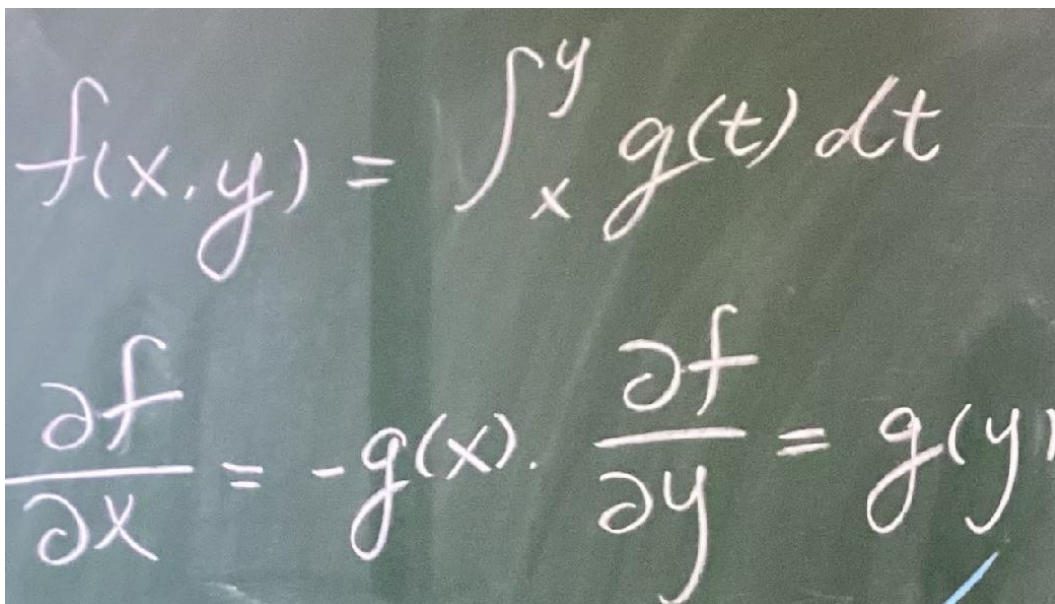
2. Section 14.3: Solutions, common mistakes and corrections:



The image shows a chalkboard with handwritten mathematical work. At the top, the function $f(x, y) = x^y = e^{y \ln x}$ is written. Below this, the partial derivative with respect to x is calculated as $\frac{\partial f}{\partial x} = e^{y \ln x} \cdot \frac{y}{x} = yx^{y-1}$, with a blue checkmark under the final result. The second partial derivative with respect to y is calculated as $\frac{\partial f}{\partial y} = e^{y \ln x} \cdot \ln x = x^y \ln x$, also with a blue checkmark under the final result.

$$f(x, y) = x^y = e^{y \ln x}$$
$$\frac{\partial f}{\partial x} = e^{y \ln x} \cdot \frac{y}{x} = yx^{y-1}$$
$$\frac{\partial f}{\partial y} = e^{y \ln x} \cdot \ln x = x^y \ln x$$

Figure 9: Solution to Section 14.3, problem 19



The image shows a chalkboard with handwritten mathematical work. The function $f(x, y) = \int_x^y g(t) dt$ is written at the top. Below this, the partial derivatives are calculated: $\frac{\partial f}{\partial x} = -g(x)$ and $\frac{\partial f}{\partial y} = g(y)$, with a blue checkmark under the final result.

$$f(x, y) = \int_x^y g(t) dt$$
$$\frac{\partial f}{\partial x} = -g(x) \quad \frac{\partial f}{\partial y} = g(y)$$

Figure 10: Solution to Section 14.3, problem 21

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{\sin h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{\sin h^4}{h^3} = 0$$

$$f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x-0}$$

$$f_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y-0}$$

Figure 11: Solution to Section 14.3, problem 60

w.r.t x : $y + 3z^2 \cdot z_x x + z^3 - 2yz_x = 0$

$(x, y, z) = (1, 1, 1)$

$$\Rightarrow 1 + 3z_x + 1 - 2z_x = 0$$

$$z_x = -2$$

Figure 12: Solution to Section 14.3, problem 65

14.3.67 (sol-2) $\frac{\partial}{\partial a}$ at both side

(b, c - const, A - func. of a) $\Rightarrow 2a = 2bc \sin A \frac{\partial A}{\partial a}$

$\Rightarrow \frac{\partial A}{\partial a} = \frac{2a}{2bc \sin A}$ - Similarly, $\frac{\partial}{\partial b}$ at both side

$d^2 = b^2 + c^2 - 2bc \cos A$

chain rule $\frac{\partial A}{\partial b} = \frac{c \cos A - b}{2bc \sin A}$

Figure 13: Solution to Section 14.3, problem 67

Solve linear equations

$\Rightarrow \frac{\partial u}{\partial x} = \frac{u \ln v}{u \ln u \ln v - u} = \frac{\ln v}{\ln u \ln v - 1} \neq$

$U_x(u, v)$ exist

$x = v \ln u \rightarrow \frac{\partial}{\partial x}$

$y = u \ln v \rightarrow \frac{\partial}{\partial x}$

$x \frac{\partial}{\partial x} = (v \ln u) \frac{\partial}{\partial x} \rightarrow$

$\rightarrow 1 = \left[\frac{\partial v}{\partial x} \right] \cdot \ln(u) + \frac{1}{u} \left[\frac{\partial u}{\partial x} \right] \cdot v$

$y \frac{\partial}{\partial x} = (u \ln v) \frac{\partial}{\partial x}$

$\rightarrow 0 = u \cdot \frac{1}{v} \left[\frac{\partial v}{\partial x} \right] + \left[\frac{\partial u}{\partial x} \right] \cdot \ln v$

Figure 14: Solution to Section 14.3, problem 69