

Brief solutions to selected problems in homework 02

1. Section 10.1: Solutions, common mistakes and corrections:

10.1.46
$$a_n = \frac{\sin^2 n}{2^n}$$
$$0 \leq \sin^2 n \leq 1$$
$$0 \leq a_n \leq \frac{1}{2^n}$$
$$\lim_{n \rightarrow \infty} 0 = \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$$
$$\therefore \lim_{n \rightarrow \infty} a_n = 0 \quad (\text{L'Hôpital})$$

Figure 1: Solution to Section 10.1, problem 46

10.1.63
$$0 < \frac{n!}{n^n} \leq \frac{1}{n}$$
$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$
$$\lim_{n \rightarrow \infty} 0 = 0$$
$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0 \quad (\text{By The Sandwich Theorem})$$

Figure 2: Solution to Section 10.1, problem 63

10.1.87

$$a_n = n - \sqrt{n^2 - n}$$

$$\lim_{n \rightarrow \infty} (n - \sqrt{n^2 - n})$$

$$= \lim_{n \rightarrow \infty} (n - \sqrt{n^2 - n}) \left(\frac{n + \sqrt{n^2 - n}}{n + \sqrt{n^2 - n}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{n^2 - (n^2 - n)}{n + \sqrt{n^2 - n}} = \lim_{n \rightarrow \infty} \frac{n}{n + \sqrt{n^2 - n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{1 - \frac{1}{n}}} = \frac{1}{2} \Rightarrow \text{conv}$$

Figure 3: Solution to Section 10.1, problem 87

10.1.89

$$a_n = \frac{1}{n} \int_1^n \frac{1}{x} dx$$

$$\lim_{n \rightarrow \infty} (\ln n - \ln 1) \frac{1}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

\Rightarrow converge

(sol-2) $\lim_{n \rightarrow \infty} \frac{\int_1^n \frac{1}{x} dx}{n} \stackrel{\#}{=} \frac{(\frac{\infty}{\infty})}{\frac{\infty}{\infty}} \stackrel{\text{FTCL}}{=} \frac{1}{1} = 0$

Figure 4: Solution to Section 10.1, problem 89

2. Section 10.2: Solutions, common mistakes and corrections:

10.2.43

$$a_n = \frac{4^n}{(2n-1)^2 (2n+1)^2}$$

$$= 5 \left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right]$$

$$S_n = 5 \left[\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{7^2} - \frac{1}{9^2} + \dots + \frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right]$$

$$= 5 \left[1 - \frac{1}{(2n+1)^2} \right]$$

$S \rightarrow 5$ as $n \rightarrow \infty$ ✓

Figure 5: Solution to Section 10.2, problem 43

10.2.63

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Both Geometric Series $|r| < 1$

$$S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{3}{4}}{1-\frac{3}{4}} = 4 \quad \text{conv.} \quad \checkmark$$

★: Make sure $\sum \left(\frac{1}{2}\right)^n$ & $\sum \left(\frac{3}{4}\right)^n$ conv. at first

Note: $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
 if $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ both conv.

Figure 6: Solution to Section 10.2, problem 63

10.2.71

$$\sum_{n=0}^{\infty} 2 \left(\frac{x-1}{2} \right)^n \quad \text{for } |r| < 1$$

$a = 2, r = \frac{x-1}{2}$

$$\lim_{n \rightarrow \infty} \frac{(x-1)^n}{2} \rightarrow 0$$

$$|x-1| < 2 \Rightarrow$$

(a) $1 < x < 3$ * $\Rightarrow \frac{1}{1-r} = \frac{1}{1 - \frac{x-1}{2}} = \frac{2}{2 - (x-1)} = \frac{2}{3-x}$

$$\text{conv. value} = \frac{a}{1-r} = \frac{2}{1 - \frac{x-1}{2}} = \frac{4}{3-x} \#$$

Figure 7: Solution to Section 10.2, problem 71

10.2.78

$$\sum_{n=0}^{\infty} (\ln x)^n = \frac{1 - (\ln x)^N}{1 - \ln x} = \frac{1 - (\ln x)^N}{1 - \ln x}$$

series converges $\Rightarrow \lim_{N \rightarrow \infty} (\ln x)^N = 0$

$$\Rightarrow |\ln x| < 1$$

$$\Rightarrow \frac{1}{e} < x < e$$

$$\sum_{n=0}^{\infty} (\ln x)^n = \lim_{N \rightarrow \infty} \frac{1 - (\ln x)^N}{1 - \ln x}$$

$$\underline{\underline{|\ln x| < 1}} \quad \underline{\underline{\frac{1}{1 - \ln x}}}$$

Figure 8: Solution to Section 10.2, problem 78

3. Section 10.3: Solutions, common mistakes and corrections:

10.3.7

$f(x)$ positive

n, n^2+4 are continuous on $[1, \infty)$

Let $f(x) = \frac{x}{x^2+4}$, $f'(x) = \frac{(4-x)^2}{(x^2+4)^2}$

$f'(x)$ is decreasing when $x > 2$

$\int_2^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{x}{x^2+4} dx$

$\int_2^{\infty} f(x) dx \text{ div.} = \lim_{b \rightarrow \infty} \frac{1}{2} \ln |x^2+4| \Big|_2^b$

$\sum_{n=2}^{\infty} f(n) \text{ div.} = \infty - \frac{1}{2} \ln 5 = \infty$

$\sum_{n=1}^{\infty} f(n) \text{ div.}$ diverge by integral test

Figure 9: Solution to Section 10.3, problem 07

10.3 28

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$$

Use Int. test.

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \infty$$

Let $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$

Let $\sqrt{x}+1 = u \Rightarrow \frac{1}{2} \frac{1}{\sqrt{x}} dx = du$

$\Rightarrow f(n) \geq 0 \quad \forall n \in \mathbb{N}$

$\textcircled{2} f$ is cont.

$\textcircled{3} f$ is decreasing

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)} = \infty$$

Figure 10: Solution to Section 10.3, problem 28

10.3.28

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$$

$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$

$$a_n = \frac{1}{\sqrt{n}(\sqrt{n}+1)} = \frac{1}{n+\sqrt{n}}$$

Set $b_n = \frac{1}{n}$ divergent $\Rightarrow \sum b_n$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{\sqrt{n}}} = \frac{1}{1+0} = 1$$

divergent #

Figure 11: Solution to Section 10.3, problem 28, another method

31) $\sum_{n=3}^{\infty} \frac{1/n}{\ln n \sqrt{\ln^2 n - 1}}$

$u = \ln n$ $f(x) = \frac{1/n}{(\ln n) \sqrt{\ln^2 n - 1}}$

$\Rightarrow du = \frac{1}{n} dn$

$\int_3^{\infty} \frac{1/n}{\ln n \sqrt{\ln^2 n - 1}}$

$= \lim_{a \rightarrow \infty} \int_{\ln 3}^a \frac{du}{\ln 3 \sqrt{|u| u^2 - 1}} = (\sec^{-1})'$

$= \sec^{-1} a - \sec^{-1} \ln 3$

$\frac{\pi}{2} - \sec^{-1} \ln 3$ (conv.)

Figure 12: Solution to Section 10.3, problem 31

10.3.41

$\sum_{n=1}^{\infty} \left(\frac{a}{n+2} - \frac{1}{n+4} \right) = \sum_{n=1}^{\infty} \frac{(a-1)n + (4a-2)}{(n+2)(n+4)}$

$a=1: \sum \frac{2}{(n+2)(n+4)}$

$\lim_{n \rightarrow \infty} \frac{2/(n+2)(n+4)}{1/n^2} = 2$ & $\sum \frac{1}{n^2}$ conv.

Limit Comparison Test (for series) $\Rightarrow \sum$ conv.

$a \neq 1: \lim_{n \rightarrow \infty} \frac{(a-1)n + (4a-2)}{(n+2)(n+4)} \cdot \frac{1/n}{1/n} = a-1 \neq 0$

Limit Comparison Test (..) ($\because \sum \frac{1}{n}$ div.)

$\Rightarrow \sum$ div.

Ans: Only $a=1$ makes \sum conv.

Figure 13: Solution to Section 10.3, problem 41

10.3.51

$$S = \sum_{n=1}^{\infty} n^{-1.1}, \quad S_k = \sum_{n=1}^k n^{-1.1}$$

Thm $\Rightarrow S - S_k = R_k < \int_k^{\infty} x^{-1.1} dx$

Goal: find k such that $R_k < 0.00001$

$$\text{LHS} = \left. \frac{-1}{0.1} x^{-0.1} \right|_k^{\infty} = 10k^{-0.1} < 10^{-5}$$

$$\Rightarrow k > 10^{60}$$

Figure 14: Solution to Section 10.3, problem 51

10.3.55

(a) Show $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$ conv. $\Leftrightarrow p > 1$

(b) What can we have about $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

(sol.) (a) $\text{LHS} = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^p} dx \stackrel{(u=\ln x)}{=} \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p} du$

($b \rightarrow \ln b$) $= \lim_{b \rightarrow \infty} \int_{\ln 2}^b \frac{1}{u^p} du = \int_{\ln 2}^{\infty} \frac{1}{u^p} du$

\star conv. $\Leftrightarrow p > 1$.

(b) $f(x) = \frac{1}{x(\ln x)^p}$ conti, positive, decreasing on $[2, \infty)$

Integral test + (a) $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ conv. $\Leftrightarrow p > 1$

Figure 15: Solution to Section 10.3, problem 55