

Brief solutions to selected problems in homework 01

1. Section 8.8: Solutions, common mistakes and corrections:

8.8.1

$\frac{1}{\sqrt{1-x^2}}$ conti. on $(0,1)$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} dx$$

let $x = \sin \theta = \lim_{b \rightarrow 1} (\sin^{-1}(b) - \sin^{-1}(0))$

$$1 - \sin^2 \theta = \cos^2 \theta, \sqrt{\cos^2 \theta} = \cos \theta$$

$$dx = \cos \theta d\theta = \sin^{-1}(1) = \frac{\pi}{2} \#$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} 1 \cdot d\theta = \theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Figure 1: Solution to Section 8.8, problem 07

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx \quad \text{①}$$

$$= \int_{-\infty}^0 \frac{2x}{(x^2+1)^2} dx + \int_0^{\infty} \frac{2x}{(x^2+1)^2} dx$$

(let $u = x^2 + 1, du = 2x dx, x(-\infty, 0, \infty) \mapsto u(\infty, 1, \infty)$)

$$= \int_{\infty}^1 \frac{du}{u^2} + \int_1^{\infty} \frac{du}{u^2}$$

$$= \lim_{a \rightarrow \infty} \int_a^1 \frac{du}{u^2} + \lim_{b \rightarrow \infty} \int_1^b \frac{du}{u^2}$$

$$= \lim_{a \rightarrow \infty} \left. -\frac{1}{u} \right|_a^1 + \lim_{b \rightarrow \infty} \left. -\frac{1}{u} \right|_1^b$$

$$= -1 - \lim_{a \rightarrow \infty} \left(-\frac{1}{a}\right) + \lim_{b \rightarrow \infty} \left(-\frac{1}{b}\right) - (-1)$$

$$= -1 - 0 + 0 + 1 = 0$$

Figure 2: Solution to Section 8.8, problem 13

$$\begin{aligned}
 & \int_0^1 x \ln x \, dx \\
 & = \lim_{b \rightarrow 0^+} \int_b^1 x \ln x \, dx \\
 & = \lim_{b \rightarrow 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_{x=b}^1 \\
 & = \frac{1}{2} \left(\lim_{b \rightarrow 0^+} b^2 \ln b \right) - \frac{1}{4} + \frac{1}{4} \lim_{b \rightarrow 0^+} b^2 \\
 & = -\frac{1}{4} \# \checkmark
 \end{aligned}$$

Figure 3: Solution to Section 8.8, problem 25

$$\begin{aligned}
 & \text{8.8.3)} \\
 & \int_{-1}^4 \frac{dx}{\sqrt{|x|}} \\
 & = \int_0^4 \frac{dx}{\sqrt{x}} + \int_{-1}^0 \frac{dx}{\sqrt{-x}} \quad \star = \lim_{b \rightarrow 0^+} \int_b^4 \frac{dx}{\sqrt{x}} \\
 & = \left[2x^{\frac{1}{2}} \right]_0^4 + \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow 0^+} \left(4 - 2b^{\frac{1}{2}} \right) \\
 & = 4 + \left[2x^{\frac{1}{2}} \right]_0^1 \\
 & = 4 + 2 \\
 & = 6 \# \checkmark
 \end{aligned}$$

Figure 4: Solution to Section 8.8, problem 31

$$(41) \int_0^{\pi} \frac{1}{\sqrt{t+\sin t}} dt$$

$$0 \leq \frac{1}{\sqrt{t+\sin t}} \leq \frac{1}{\sqrt{t}}$$

$$\int_0^{\pi} \frac{1}{\sqrt{t}} dt = \lim_{a \rightarrow 0^+} \int_a^{\pi} \frac{1}{\sqrt{t}} dt$$

$$= \lim_{a \rightarrow 0^+} 2\sqrt{t} \Big|_a^{\pi}$$

$$= \lim_{a \rightarrow 0^+} (2\sqrt{\pi} - 2\sqrt{a})$$

$$= 2\sqrt{\pi} < \infty$$

$$\Rightarrow \int_0^{\pi} \frac{1}{\sqrt{t+\sin t}} dt$$

converges

Figure 5: Solution to Section 8.8, problem 41

$$\begin{aligned}
& \int_{-1}^1 \ln|x| dx \\
&= \int_0^1 \ln|x| dx + \int_{-1}^0 \ln|x| dx \\
&= \lim_{a \rightarrow 0^+} \int_a^1 \ln|x| dx + \lim_{b \rightarrow 0^-} \int_{-1}^b \ln|x| dx \\
&= \lim_{a \rightarrow 0^+} [x \cdot \ln x - x]_a^1 + \lim_{b \rightarrow 0^-} [x \cdot \ln(-x) - x]_{-1}^b \\
&= -1 + 1 = 0, \therefore \text{converges.}
\end{aligned}$$

Figure 6: Solution to Section 8.8, problem 45

sec 8.8

(55)

$$\int_{\pi}^{\infty} \frac{2 + \cos x}{x} dx$$

$$\boxed{\frac{2 + \cos x}{x} = \frac{1}{x}} \text{ for } x \in \mathbb{R}, x > 0$$

$$\begin{aligned}
\int_{\pi}^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_{\pi}^b \frac{1}{x} dx \\
&= \lim_{b \rightarrow \infty} \ln x \Big|_{\pi}^b \\
&= \lim_{b \rightarrow \infty} \ln b - \ln \pi = \infty.
\end{aligned}$$

$\therefore \int_{\pi}^{\infty} \frac{1}{x} dx$ diverges & $\frac{2 + \cos x}{x} = \frac{1}{x}$ for $x \in \mathbb{R}$

$\therefore \int_{\pi}^{\infty} \frac{2 + \cos x}{x} dx$ diverges.

Figure 7: Solution to Section 8.8, problem 55

65

a) $\int_1^2 \frac{dx}{x(\ln x)^p} = \int_0^{\ln 2} \frac{du}{u^p}$

$\begin{cases} \ln x = u \\ \frac{1}{x} dx = du \end{cases} = \begin{cases} \text{con} & \text{if } p < 1 \\ \text{o.w.} & \end{cases}$

b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

$= \int_{\ln 2}^{\infty} \frac{du}{u^p} = \begin{cases} \text{con} & \text{if } p > 1 \\ \text{o.w.} & \end{cases}$

Figure 8: Solution to Section 8.8, problem 65

8.8.66

$\int_0^{\infty} \frac{2x dx}{x^2+1} = \lim_{b \rightarrow \infty} \int_0^b \frac{2x dx}{x^2+1} = \lim_{b \rightarrow \infty} \ln(x^2+1) \Big|_0^b$

$= \lim_{b \rightarrow \infty} \ln(b^2+1) - \ln 1 = +\infty$ div

$\int_{-\infty}^{\infty} \frac{2x dx}{x^2+1} = \int_{-\infty}^0 \frac{2x dx}{x^2+1} + \int_0^{\infty} \frac{2x dx}{x^2+1} \Rightarrow \text{div}$

$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{2x dx}{x^2+1} = \lim_{b \rightarrow \infty} \ln(x^2+1) \Big|_{-b}^b = \lim_{b \rightarrow \infty} \ln(b^2+1) - \ln((-b)^2+1)$

$= 0$

Figure 9: Solution to Section 8.8, problem 66

2. Section 8.8: Homework 01, problem 4:

4.

$$\int_0^{\infty} \frac{1}{e^{x^{0.5} \ln x}} dx$$

1^o Let $g(x) = \frac{1}{x}$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^{0.2}}{\ln x} = \infty$$

2^o $\int_0^{\infty} \frac{1}{e^{x^{0.5} \ln x}} dx$ (div) $\therefore \int_0^{\infty} \frac{1}{x} dx$ (div) #

Figure 10: Solution to Section homework 01, problem 4

3. Section 8.8: Homework 01, problem 5:

5.

(note: $\lim_{x \rightarrow 0^+} x^2 \ln x = 0$)

$$\int_0^{\frac{1}{e}} \frac{1}{x^2 |\ln x|} dx = \lim_{t \rightarrow 0^+} \int_t^{\frac{1}{e}} \frac{1}{x^2 (-\ln x)} dx$$

$$\frac{1}{x^2 (-\ln x)} = \frac{1}{x^2 \ln \frac{1}{x}}$$

For $0 < x < \frac{1}{e}$, $\ln \frac{1}{x} < \frac{1}{x}$, so $\frac{1}{x^2 \ln \frac{1}{x}} > \frac{1}{x^2 \cdot \frac{1}{x}} = \frac{1}{x}$

$e < \frac{1}{x} < \infty$

$$\lim_{t \rightarrow 0^+} \int_t^{\frac{1}{e}} \frac{1}{x} dx$$

By Direct Comparison

div. /

Figure 11: Solution to Section homework 01, problem 5

(sol-2)

Compare with $\int_0^{\frac{1}{e}} \frac{1}{x^{\frac{3}{2}}} dx$

$$\lim_{x \rightarrow 0^+} \frac{1/x^2 |\ln x|}{1/x^{\frac{3}{2}}} = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x} |\ln x|} = \infty$$

$$\int_0^{\frac{1}{e}} \frac{1}{x^{\frac{3}{2}}} dx \text{ div.}$$

limit comparison

$$\rightarrow \int_0^{\frac{1}{e}} \frac{1}{x^2 |\ln x|} dx \text{ div.}$$

Figure 12: Solution to Section homework 01, problem 5