1. (15)

$$\begin{cases} \left(\frac{\partial w}{\partial x}\right)_y = 0 \\ \left(\frac{\partial w}{\partial y}\right)_x = 0 \end{cases} \Rightarrow \begin{cases} 0 = 1 + 2\left(\frac{\partial z}{\partial x}\right)_y \\ 0 = 2 + 2\left(\frac{\partial z}{\partial y}\right)_x \end{cases} \quad (3) \quad \Rightarrow \begin{cases} 0 = 1 + 2\,\frac{-2x}{2z} \\ 0 = 2 + 2\,\frac{-2y}{2z} \end{cases} \Rightarrow \begin{cases} \frac{x}{z} = \frac{1}{2} \\ \frac{y}{z} = 1. \end{cases}$$

$\because x^2+y^2+z^2=9 \quad \therefore (x,y,z) = P_1(1,2,2),\ P_2(-1,-2,-2) \Rightarrow$ critical points. (2)

$$(w_{xx} =)\ \left(\frac{\partial^2 w}{\partial x^2}\right)_y = \left(\frac{\partial}{\partial x}\right)_y \left(1 - 2\frac{x}{z}\right) = (-2)\,\frac{z - \left(\frac{\partial z}{\partial x}\right)_y \cdot x}{z^2} = (-2)\,\frac{z^2+x^2}{z^3}$$

$$(w_{yy} =)\ \left(\frac{\partial^2 w}{\partial y^2}\right)_x = \left(\frac{\partial}{\partial y}\right)_x \left(2 - 2\frac{y}{z}\right) = (-2)\,\frac{z - \left(\frac{\partial z}{\partial y}\right)_x \cdot y}{z^2} = (-2)\cdot\frac{z^2+y^2}{z^3}$$

$$(w_{xy} =)\ \left(\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)_y\right)_x = \left(\frac{\partial}{\partial y}\right)_x \left(1 - 2\frac{x}{z}\right) = -2x \cdot \frac{-\left(\frac{\partial z}{\partial y}\right)_x}{z^2} = (-2)\cdot\frac{xy}{z^3}$$

$$\Rightarrow \left.\begin{cases} w_{xx}(P_1) = \frac{-5}{4},\ w_{yy}(P_1) = -2,\ w_{xy}(P_1) = \frac{-1}{2}. \\ w_{xx}(P_2) = \frac{5}{4},\ w_{yy}(P_2) = 2,\ w_{xy}(P_2) = \frac{1}{2}. \end{cases}\right\} \quad (4)$$

Know 2nd derivative test (4)

$\because D(P_1) = \left(\frac{-5}{4}\right)(-2) - \left(\frac{-1}{2}\right)^2 > 0,\ w_{xx}(P_1) < 0 \Rightarrow$ local max. at P_1. (1)

$\because D(P_2) = \frac{5}{4}\cdot 2 - \left(\frac{1}{2}\right)^2 > 0,\ w_{xx}(P_2) > 0 \Rightarrow$ local min. at P_2 (1)

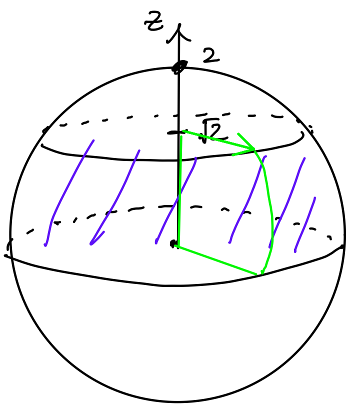
Remark. (1) 用 Lagrange Multiplier or 其他方式算 不给分.

(2) 用 $\left(\frac{\partial w}{\partial z}\right)_y = \left(\frac{\partial w}{\partial y}\right)_z = 0$ or $\left(\frac{\partial w}{\partial x}\right)_z = \left(\frac{\partial w}{\partial z}\right)_x = 0$ 算：配分同理.

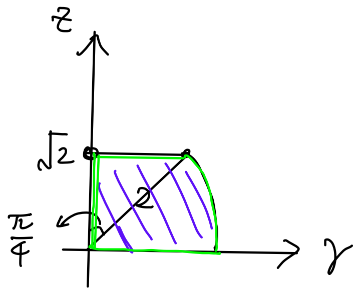
2. (15)

$r \le \sqrt{4-z^2} \Rightarrow \rho^2 = r^2+z^2 \le 4,\ \rho \le 2$. Combining $0 \le z \le \sqrt{2}$, $0 \le \theta \le 2\pi$.

we have :



θ-cross section $\longrightarrow$ (for any $0 \le \theta \le 2\pi$)



By the cross section,

1° $0 \le \phi \le \frac{\pi}{4}$: $\because z=\sqrt{2} \Leftrightarrow \rho = \frac{\sqrt{2}}{\cos\phi}$ $\therefore$ $0 \le \rho \le \frac{\sqrt{2}}{\cos\phi}$.

2° $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$: $0 \le \rho \le 2$.

5

$$\Rightarrow = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\sqrt{2}}{\cos\phi}} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \overset{5}{=} \frac{(a\sqrt{2}}{3}\pi$$

10

沒有乘以 $\rho^2\sin\phi$ -3，ρ, ϕ, θ 範圍寫錯各 -3，扣至 0 為止。

只有畫出 cross section：$+3$

<u>Remark</u>. Equivalently, $= \left[\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} - \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{\sqrt{2}}{\cos\phi}}^{2}\right] \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.

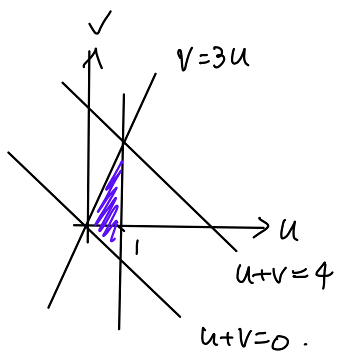
$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{\frac{\pi}{2}} \rho^2\sin\phi\, d\phi\, d\rho\, d\theta + \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_{\cos^{-1}\left(\frac{\sqrt{2}}{\rho}\right)}^{\frac{\pi}{2}} \rho^2\sin\phi\, d\phi\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{\frac{\pi}{2}} \rho^2\sin\phi\, d\phi\, d\rho\, d\theta - \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_0^{\cos^{-1}\left(\frac{\sqrt{2}}{\rho}\right)} \rho^2\sin\phi\, d\phi\, d\rho\, d\theta.$$

3. (15)

$$\begin{array}{l} u = x+y \\ v = 3x-y \end{array} \Rightarrow \begin{array}{l} x = \frac{u+v}{4} \\ y = \frac{3u-v}{4} \end{array} \;(4) \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{-1}{4} \end{vmatrix} = \frac{-1}{4} \;(4)$$

$0 \le x \le 1 \Leftrightarrow 0 \le u+v \le 4$, $0 \le y \le 1-x \Leftrightarrow v \le 3u$ & $u \le 1$.



$$\Rightarrow \underbrace{\int_0^1\int_{-u}^{3u}}_{4} \frac{v^2}{\sqrt{u}} \cdot \left|\frac{-1}{4}\right| dv\, du \overset{3}{=} \frac{2}{3}$$ #

4. 15

Let $x=3u,\ y=2v,\ z=w \Rightarrow \{u^2+v^2+w^2 \le 1\}$; $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 6.$ 4

$$\iiint_D |xyz|\, dV = \iiint_{\{u^2+v^2+w^2\le 1\}} 6\,|uvw| \cdot 6\, dV$$ 2

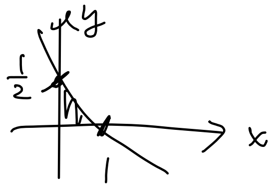
(symmetry) $$= 36\cdot 8 \iiint_{\left\{\substack{u^2+v^2+w^2\le 1 \\ u,v,w\ge 0}\right\}} uvw\, dV.$$ 3

(spherical integral) $$= 288 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta\,\rho\cos\phi\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$ 3

$$= 288\left(\int_0^1 \rho^5\, d\rho\right)\left(\int_0^{\frac{\pi}{2}} \sin^3\phi\cos\phi\, d\phi\right)\left(\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta\right) = 6$$ # 3

5. 20

$R = \{0\le x\le 1,\ 0\le y\le 2^{-x}-\frac{1}{2}\}.$



(a) 8 Let C be the boundary of R (counterclockwise).

$\Rightarrow$ (i) $\oint_C F\cdot T\, ds = \oint_C M\,dx + N\,dy = \iint_R N_x - M_y\, dA.$ 4

(ii) $\oint_C F\cdot n\, ds = \oint_C M\,dy - N\,dx = \iint_R M_x + N_y\, dA.$ 4

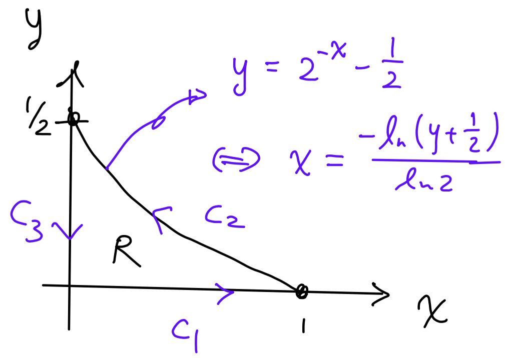
(b) 12 Prove (i) (配分 of (ii) is similar)

Let $C = C_1 \cup C_2 \cup C_3$, $f(x) = 2^{-x}-\frac{1}{2}$, $g(y) = -\dfrac{\ln(y+\frac{1}{2})}{\ln 2}$

$C_1 = \{(t,0) \mid 0\le t\le 1\}$

$C_2 = \{(g(t),t) \mid 0\le t\le \frac{1}{2}\}.$

$C_3 = \{(0,\frac{1}{2}-t) \mid 0\le t\le \frac{1}{2}\}.$ 2



$\Rightarrow \oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}.$

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 M(t,0)\,dt + 0.$$ 1

$$\int_{C_3} M\,dx + N\,dy = \int_0^{\frac{1}{2}} N\left(0, \tfrac{1}{2}-t\right)(-1)\,dt \overset{u=\frac{1}{2}-t}{=} -\int_0^{\frac{1}{2}} N(0,u)\,du.$$ 2

$$\int_{C_2} M\,dx + N\,dy = \int_0^{\frac{1}{2}} M(g(t),t)\,g'(t) + N(g(t),t)\,dt.$$ 2

$$\begin{pmatrix} u = g(t) \\ \Rightarrow t = f(u) \end{pmatrix} 2 \quad = \int_1^0 M(u, f(u))\,du + \int_0^{\frac{1}{2}} N(g(t),t)\,dt.$$

$$\Rightarrow \text{LHS} = \int_0^1 M(x,0) - M(x,f(x))\,dx + \int_0^{\frac{1}{2}} N(g(y),y) - N(0,y)\,dy$$

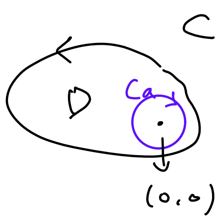
$$2 \quad = \int_0^1 \int_0^{f(x)} -M_y(x,y)\,dy\,dx + \int_0^{\frac{1}{2}} \int_0^{g(y)} N_x(x,y)\,dx\,dy$$

$$= \iint_R N_x - M_y\,dA. \quad \#.$$

Set specific F and correct : +4

6. 20

Let $C_a = \{x^2+y^2 = a^2\}$, clockwise, such that C_a is contained in the region enclosed by C. Let D be the region between them. Let $F = \langle M, N\rangle$. By Green's Theorem,



$$\oint_C F\cdot T\,ds + \oint_{C_a} F\cdot T\,ds = \iint_D N_x - M_y\,dA$$ 6

$$\left(\text{resp. } \oint_C F\cdot n\,ds + \oint_{C_a} F\cdot n\,ds = \iint_D M_x + N_y\,dA\right).$$

Now $M_x = -N_y = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ 3, $N_x = M_y = \dfrac{-2xy}{(x^2+y^2)^2}$ 3 $\Rightarrow$ $N_x - M_y = 0$, $M_x + N_y = 0$.

$$\Rightarrow \oint_C F\cdot T\,ds = \oint_{-C_a} F\cdot T\,ds, \quad \oint_C F\cdot n\,ds = \oint_{-C_a} F\cdot n\,ds.$$

$-C_a$ is counterclockwise $\Rightarrow$ let $-C_a : \{(a\cos t, a\sin t) \mid 0 \le t \le 2\pi\}$.

Wrong Ca : -3

$r(t) = (a\cos t, a\sin t) \Rightarrow T(t) = (-\sin t, \cos t), \quad n(t) = (\cos t, \sin t).$

$$\Rightarrow \oint_{-C_a} F\cdot T\, ds \overset{2}{=} \int_0^{2\pi} \left\langle \frac{\cos t}{a}, \frac{\sin t}{a} \right\rangle \cdot \langle -\sin t, \cos t\rangle \cdot \underbrace{a\, dt}_{= ds} \overset{2}{=} 0 \quad \#$$

$$\oint_{-C_a} F\cdot n\, ds \overset{2}{=} \int_0^{2\pi} \left\langle \frac{\cos t}{a}, \frac{\sin t}{a} \right\rangle \cdot \langle \cos t, \sin t\rangle \cdot a\, dt \overset{2}{=} 2\pi \quad \#$$

(sol-2).

F is conservative $\left(f(x,y) = \frac{1}{2}\ln(x^2+y^2)\right) \Rightarrow \oint_C F\cdot T\, ds = 0.$ 10

(sol-3)

def. [illegible] $\Rightarrow \oint_C F\cdot T\, ds = 0$ 10