Ratio Test and Root Test

Thm (Ratio Test)

If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$,

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Longrightarrow$ inconclusive.

Thm $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

pf: $-|a_n| \le a_n \le |a_n|$, $0 \le a_n + |a_n| \le 2|a_n|$

$\sum |a_n| < \infty \Rightarrow \sum 2|a_n| < \infty \Rightarrow \sum a_n + |a_n| < \infty$

$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| = \text{conv} - \text{conv} = \text{conv.}$

Thm (Root Test)

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Rightarrow$ inconclusive.

Eg 1 $\begin{cases} \sum \frac{1}{n} = \infty & (\rho = 1) \\ \sum \frac{1}{n^2} < \infty & (\rho = 1) \end{cases}$

case ⓒ for both ratio test and root test.

$=\infty$

Eg2 $a_n = (-1)^{n+1}\frac{1}{n}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k} \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \quad \text{conv}$$

(∵ Compare with $\sum_{k=1}^{\infty} \frac{1}{k^2}$)

∴ ($\sum |a_n| = \infty \not\Rightarrow \sum a_n$ div)

Eg 3

(a) $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Sol: Ratio Test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{2^{n+1}+5}{3\cdot(2^n+5)}$$

$$= \lim_{n\to\infty} \frac{2+5\cdot 2^{-n}}{3\cdot(1+5\cdot 2^{-n})} = \frac{2}{3}$$

$\rho < 1 \Rightarrow$ convergent.

Root test:

$$\lim |a_n|^{\frac{1}{n}} = \lim \left(\frac{2^n(1+5.2^{-n})}{3^n}\right)^{\frac{1}{n}}$$

$$= \frac{2}{3} < 1$$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Sol. Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4 > 1$$

Ratio test $\Rightarrow$ divergent

ⓒ $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

Sol: Ratio Test

$\rho = 1$ (See ⓑ)

Ratio Test is inconclusive

However,

$$\frac{a_{n+1}}{a_n} \overset{ⓑ}{=} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$= \frac{(2n+2)\cancel{(2n+2)}}{(2n+1)\cancel{(2n+2)}} > 1$$

$\Rightarrow a_{n+1} > a_n > 0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ div!

(d) $\sum_{n=1}^{\infty} \frac{n^2}{2^n} \xRightarrow[\text{Root}]{\text{Ratio}} \rho = \frac{1}{2}$. conv.

(e) $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n \xRightarrow{\text{Root}} \rho = 0$ conv.

(f) $a_n = \begin{cases} \frac{n}{2^n} & n \text{ is odd} \\ \frac{1}{2^n} & n \text{ is even} \end{cases}$

$= \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{16}, \frac{5}{32}, \frac{1}{64}, \frac{7}{128}, \frac{1}{256}, \cdots$

$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2} \cdot \frac{1}{n} & n \text{ is odd.} \\ \frac{1}{2}(n+1) & n \text{ is even} \end{cases}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist : inconclusive.

However $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{2} \Rightarrow$ convergent.

Proof (of the Ratio Test
The Root test is similar)

(1) $0 \leq \rho < 1$, $\left(\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \right)$

Take $r = \frac{1+\rho}{2} < 1$

Since $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$

Take $\varepsilon = r - \rho > 0$, there exists a corresponding $N \in \mathbb{N}$ such that

"$n > N \implies \left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon$"

$\left(\implies \left| \frac{a_{n+1}}{a_n} \right| < \underbrace{\rho + \varepsilon}_{= r} \right)$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n|$$

$$= \sum_{n=1}^{N}|a_n| + |a_{N+1}| + |a_{N+2}| + \cdots$$

$$\leq (\quad '' \quad) + |a_N|r + |a_N|r^2 + \cdots\cdots$$

$$= \left(\sum_{n=1}^{N}|a_n|\right) + \text{convergent Geometric Series} < \infty$$

(2) $\rho > 1$, define $r = \frac{1+\rho}{2} > 1$

let $\varepsilon = r - \rho > 0 \Rightarrow \exists N \in \mathbb{N}$

such that

"$n > N \Rightarrow \left|\left|\frac{a_{N+1}}{a_N}\right| - \rho\right| < \varepsilon$"

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > \rho - \varepsilon = r > 1$.

$\Rightarrow \lim |a_n| \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Alternating Series
(Leibnitz test)

If (1) $u_n > 0$

(2) $u_n \geq u_{n+1}$

(for all $n \geq N$)

(3) $\lim_{n\to\infty} u_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges

Eg1: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges

Sol $u_n = \frac{1}{n} > 0$.

$u_n > u_{n+1}$. $\lim u_n = 0$

$\therefore$ Leibnitz test $\Rightarrow$ converges

Def (1) $\sum a_n$ converges <u>absolutely</u>

if $\sum |a_n| < \infty$

(2) $\sum a_n$ converges <u>conditionally</u>

if $\begin{cases} \sum a_n \text{ converges} \\ \sum |a_n| = \infty \end{cases}$

Eg2 (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ $\quad p > 0$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv.} & p > 1 \\ \text{div} & 0 \le p \le 1 \end{cases}$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges $\begin{matrix} \text{abs.} & p > 1 \\ \text{cond.} & 0 \le p \le 1 \end{matrix}$

Eg 3 $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$. conv abs? cond.?

Sol: $\sum (-1)^{n+1} u_n$ converges (Leibnitz Test)

Does $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converge?

Integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{x=2}^{\infty} \frac{1}{\sqrt{\ln x}}\,d\ln x$$

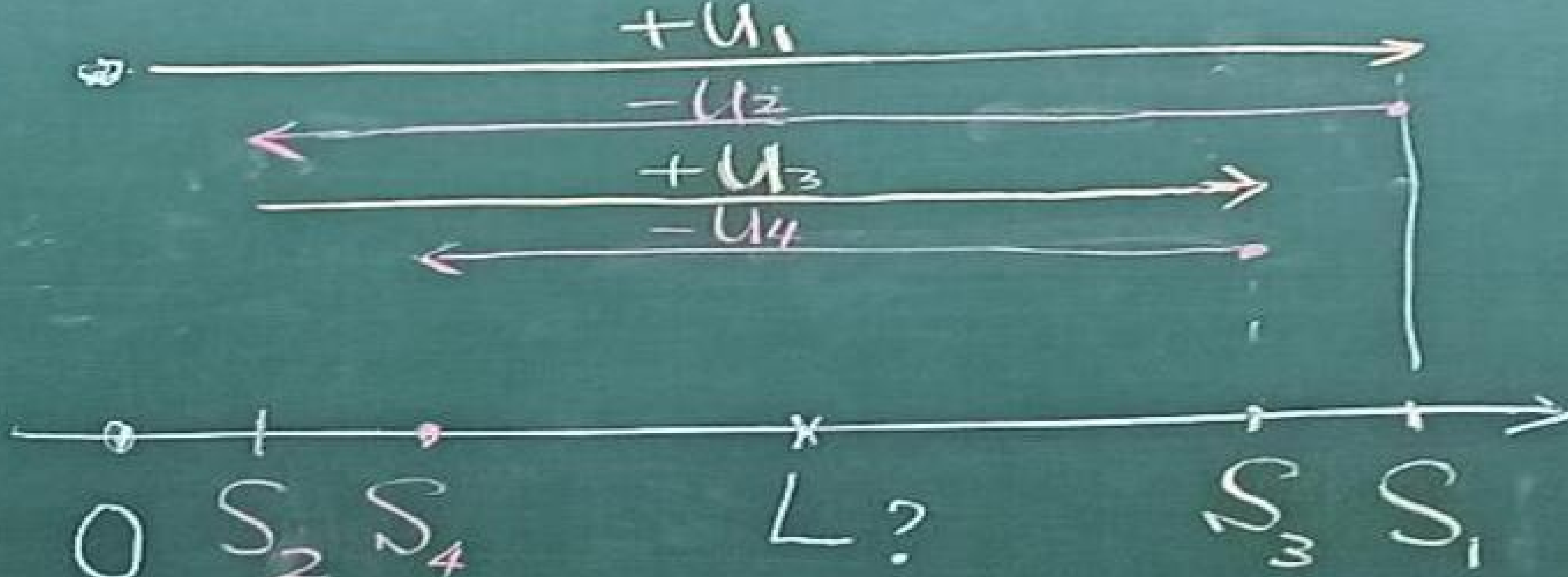
$$= \int_{y=\ln 2}^{\infty} \frac{1}{\sqrt{y}}\,dy \quad (y = \ln x)$$

$= $ "$p = \frac{1}{2}$" $=$ divergent.

Ans: $\sum \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$ converges cond.

Pf of Leibnitz test

Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} u_k$



$\Rightarrow 0 < S_2 < S_4 < \cdots\cdots < S_3 < S_1$

$\therefore \{S_2, S_4, \ldots \ldots S_{2k}, \ldots\ldots\}$

is an <u>increasing</u> sequence.

and <u>bounded above</u> ($S_{2k} < S_1$)

From the Monotone Sequence Thm (Section 10.1. Thm 6)

$\lim_{k\to\infty} S_{2k} = L$ for some $L \in \mathbb{R}$

Moreover $S_{2k+1} = S_{2k} + u_{2k+1} \xrightarrow[k\to\infty]{} L + 0$

$\therefore \lim_{n\to\infty} S_n = L$

Remark: (1) We assumed for simplicity $u_1 \geq u_2 \geq u_3 \geq \cdots\cdots$ (i.e. decreasing for all n)

(2) $S_{2k} < L < S_{2\ell-1}$ $\forall k, \ell \in \mathbb{N}$

$\Rightarrow$ $0 < L - S_{2k} < u_{2k+1}$

$0 < S_{2\ell-1} - L < u_{2\ell}$

(error estimate of partial sum)