

The Integral Test

Thm If $a_n > 0$, $a_n = f(n)$ where $f(x)$ is positive, continuous and decreasing for all $x \geq N$, $N \in \mathbb{N}$ (*)

Then $\sum_{n=N}^{\infty} a_n < \infty \iff \int_N^{\infty} f(x) dx < \infty$

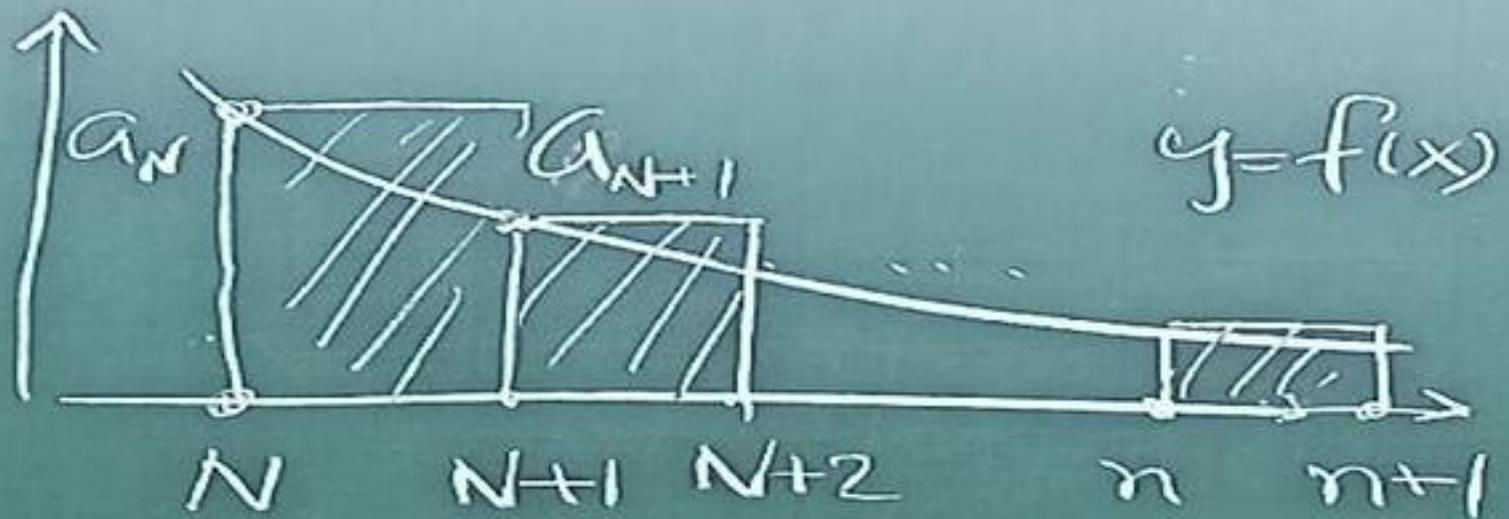
($a_n > 0$, $\sum_{n=N}^{\infty} a_n < \infty \iff \sum_{n=N}^{\infty} a_n$ conv. *)

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges?

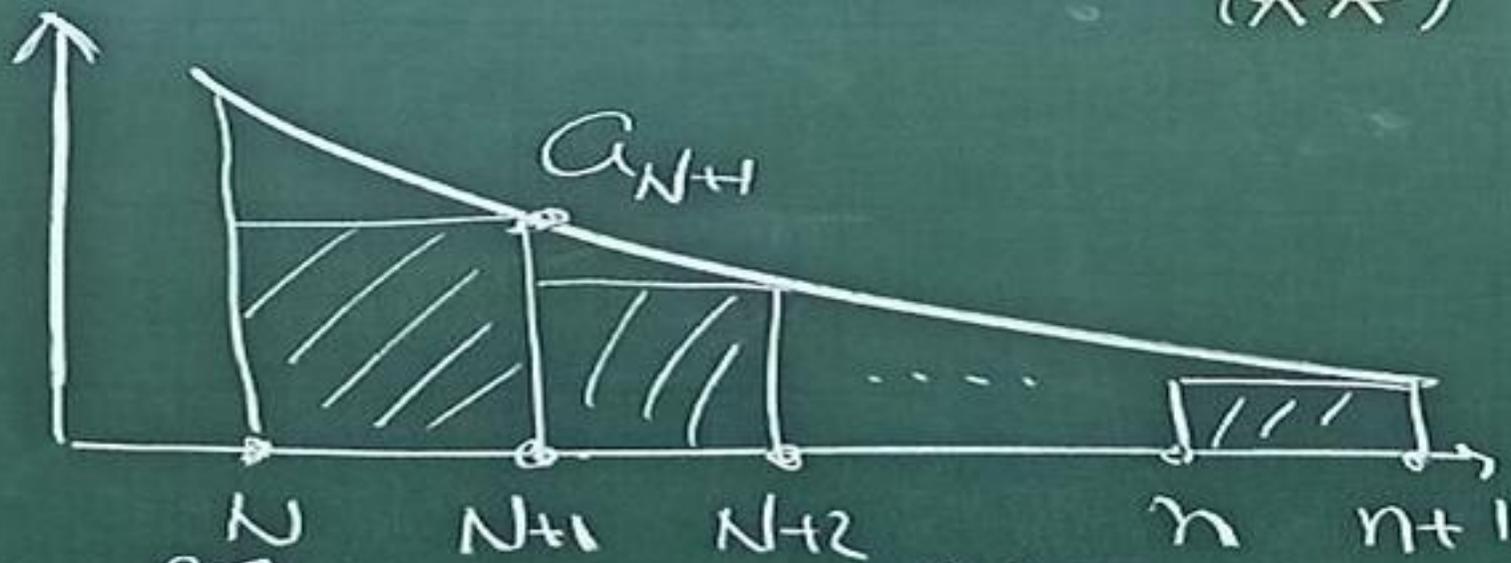
Sol $p \leq 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^p} \neq 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$

$p > 0 \Rightarrow$ Compare with $\int_1^{\infty} \frac{1}{x^p} dx$ $\begin{cases} < \infty & p > 1 \\ = \infty & 0 < p \leq 1 \end{cases}$

pf.



$$\Rightarrow \sum_{n=N}^{\infty} a_n \geq \int_N^{\infty} f(x) dx \quad (**)$$



$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x) dx \quad (***)$$

$$\int_N^{\infty} f(x) dx \leq \sum_{n=N}^{\infty} a_n \leq a_N + \int_N^{\infty} f(x) dx \quad (**)$$

(a_N is a fixed number)

$$\therefore \sum_{n=N}^{\infty} f(x) dx < \infty \Leftrightarrow \sum_{n=N}^{\infty} a_n < \infty$$

R_M . If (*) holds, and

$$\sum_{n=N}^{\infty} a_n = L < \infty, \text{ then}$$

we can estimate R_M ($M > N$)

$$R_M = \sum_{n=N}^{\infty} a_n - \sum_{n=N}^M a_n \left(= \sum_{n=M+1}^{\infty} a_n \right)$$

$$\int_{M+1}^{\infty} f(x) dx \leq \sum_{n=M+1}^{\infty} a_n \leq \int_M^{\infty} f(x) dx$$

$$\begin{array}{ccc} (N \rightarrow M+1 \text{ in } (**)) & \downarrow & (N \rightarrow M \text{ in } (***)) \\ & & = R_M \end{array}$$

$$\text{Eg 1. } \sum_{n=1}^{\infty} n^{-p} \begin{cases} = \infty, & 0 < p \leq 1 \\ < \infty, & p > 1 \end{cases}$$

$$\text{Eg 2. } \sum_{n=0}^{\infty} \frac{1}{1+n^2} < \infty$$

$$\text{Sol. } \int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx$$

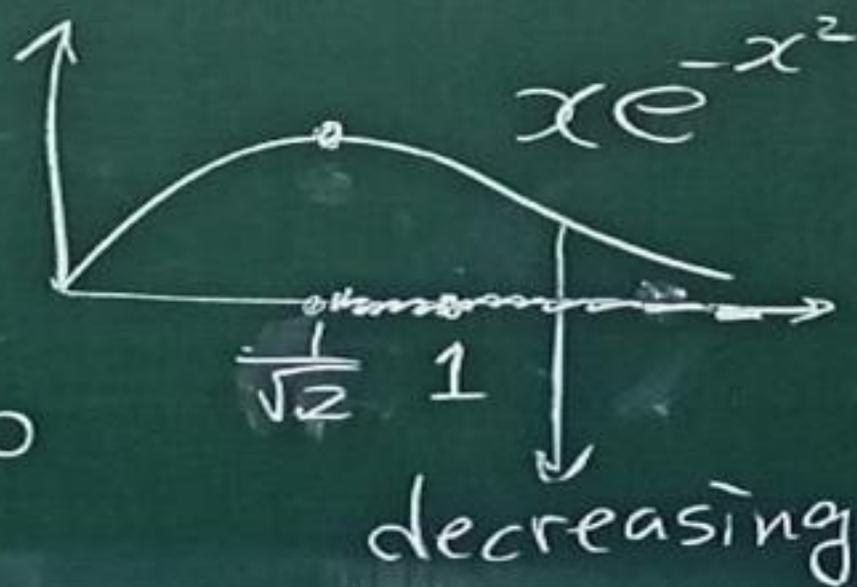
$$= \lim_{b \rightarrow \infty} \tan^{-1} b - \tan^{-1} 0 = \frac{\pi}{2} < \infty$$

$$\text{Eg 3. } \sum_{n=1}^{\infty} n e^{-n^2} < \infty$$

$$\text{Sol. } \int_0^{\infty} x e^{-x^2} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-x^2} dx < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} n e^{-n^2} < \infty$$



$$\text{Ex 4 } \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} \leq \infty ?$$

$$\text{Sol: } \int_2^{\infty} \frac{1}{2^{\ln x}} dx = ?$$

$$y = \ln x, \quad x = e^y, \quad dx = e^y dy$$

$$= \int_{y=\ln 2}^{\infty} 2^{-y} e^y dy$$

$$= \int_{\ln 2}^{\infty} \left(\frac{e}{2}\right)^y dy = \infty$$

$\frac{e}{2} > 1$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{2^{\ln n}} = \infty$$

Direct Comparison Test

If $0 < d_n \leq a_n \leq c_n$

for all $n > N$, then

① $\sum_{n=1}^{\infty} c_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

② $\sum_{n=1}^{\infty} d_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n = \infty$

Ex 5 $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

Sol $\frac{5}{5n-1} > \frac{1}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n-1} = \infty$

Limit Comparison Test ($\sum_{n=1}^{\infty} a_n < \infty$)

If $a_n > 0, b_n > 0 \quad \forall n \geq N$

(i) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c, 0 < c < \infty$

Then $\sum_{n=1}^{\infty} b_n < \infty \iff \sum_{n=1}^{\infty} a_n < \infty$
($= \infty$) ($= \infty$)

(ii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$

Then $\sum_{n=1}^{\infty} b_n < \infty \implies \sum_{n=1}^{\infty} a_n < \infty$

(iii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = +\infty$

Then $\sum_{n=1}^{\infty} b_n = \infty \implies \sum_{n=1}^{\infty} a_n = \infty$

Ex 6 $\sum_{n=1}^{\infty} \frac{5}{5n+1}$

Sol $\frac{5}{5n+1} < \frac{1}{n}$

\therefore Direct Comparison Test is inconclusive,

Limit Comparison Test

$$\lim_{n \rightarrow \infty} \frac{\frac{5}{5n+1}}{\frac{1}{n}} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{5}{5n+1} = \infty$$

Ex 7: $\sum_{n=0}^{\infty} \frac{1}{n!} < \infty$ ($0! \equiv 1$)

Sol: $1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$
 $< 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots < \infty$

Eg 8. $\sum a_n$ $\sum b_n$ Thm Result

(a) $\sum_{n=1}^{\infty} \frac{2n}{(n+1)^2}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ $\frac{11}{(c=1)}$ (i) div.

(b) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\frac{11}{(c=1)}$ (i) conv.

(c) $\sum_{n=1}^{\infty} \frac{1+n \ln n}{n^2+5}$ $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\frac{11}{(c=1)}$ (i) div

$\sum_{n=1}^{\infty} \frac{\ln n}{n} < \infty$?
 $= \infty$?

Method 1. $\sum \frac{\ln n}{n} > \sum \frac{1}{n} = \infty$

Method 2. $\int_3^{\infty} \frac{\ln x}{x} dx = \infty \Rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{n} = \infty$

($\frac{\ln x}{x}$ is decreasing for $x > e = 2.718...$)

	$\sum a_n$	$\sum b_n$	Thm	Result
(d)	$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$	$\sum_{n=1}^{\infty} \frac{1}{n}$	11 (i) (C=1)	<u>div</u>

(e)	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$	$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$	11 (i) (C=1)	<u>conv</u>
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(f)	$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$	$\sum_{n=1}^{\infty} \frac{n^{0.2}}{n^{3/2}}$	11 (ii) (C=0)	<u>conv</u>
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$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{(\ln n)^2}{n^{0.1}} \right)$$

$$= \left(\lim_{n \rightarrow \infty} \frac{\ln n}{n^{0.1}} \right)^2 \stackrel{\text{Hospital}}{=} \left(\lim_{n \rightarrow \infty} \frac{1/n}{0.1 n^{-0.9}} \right)^2$$

$$= 0$$

Eg 8 $\sum a_n \gtrsim \sum b_n$ Thm Result

(g) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^2}\right) \gtrsim \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\| (i)$ conv
(C=1)

$$\therefore \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) \left(\frac{1}{\cos \theta} \right) = 1$$

(h) $\sum_{n=1}^{\infty} \frac{1}{n \sqrt{n}} \gtrsim \sum_{n=1}^{\infty} \frac{1}{n}$ $\| (i)$ div
(C=1)

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

$\sum a_n$ $\sum b_n$ Thm Res.

② $\sum_{n=1}^{\infty} \sqrt{\frac{\ln n}{n}}$ $\sum \frac{1}{\sqrt{n}}$ 10 div.

① $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \ln n}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ || (iii) div.
($c = +\infty$)

$p = \frac{1}{2} \rightarrow$ inconclusive.

$p = 1, \frac{3}{4}, 0.9, 0.51, \dots$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{\frac{\ln n}{n}}}{\frac{1}{n}}$$

(take $p=1$) $= \lim_{n \rightarrow \infty} \left(\frac{n}{\ln n} \right)^{\frac{1}{2}} = \infty$

Since $\sum b_n = \sum \frac{1}{n} = \infty$
 $\Rightarrow \sum a_n = \infty$