# Remark on Definition of Differentiability (v04)

The following definitions are equivalent (i.e. the same, even though they look different)

**Definition 1**: $f(x,y)$ is differentiable at $(x_0,y_0)$ if

$\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$ exist and

$$f(x,y) = L(x,y)+\varepsilon_1\cdot(x-x_0)+\varepsilon_2\cdot(y-y_0),\ \lim_{(x,y)\to(x_0,y_0)}\varepsilon_1 = \lim_{(x,y)\to(x_0,y_0)}\varepsilon_2 = 0, \quad (1)$$

(other equivalent definitions include:

$$f(x,y) = L(x,y) + \varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}, \qquad \lim_{(x,y)\to(x_0,y_0)}\varepsilon = 0, \qquad (2)$$

or

$$\Delta z = \partial_x f(x_0,y_0)\,\Delta x + \partial_y f(x_0,y_0)\,\Delta y + \varepsilon_1\cdot\Delta x + \varepsilon_2\cdot\Delta y \quad \text{(textbook version)} \quad (3)$$

or

$$\Delta z = \partial_x f(x_0,y_0)\,\Delta x + \partial_y f(x_0,y_0)\,\Delta y + \varepsilon\cdot\sqrt{(\Delta x)^2+(\Delta y)^2} \quad ) \qquad (4)$$

where

$$L(x,y) = f(x_0,y_0) + \partial_x f(x_0,y_0)\cdot(x-x_0) + \partial_y f(x_0,y_0)\cdot(y-y_0), \qquad (5)$$

and

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta z = f(x,y) - f(x_0,y_0). \qquad (6)$$

**Remark 1**:

In view of the identity (see homework 07, problem 2 for hint of proof)

$$\varepsilon_1\cdot(x-x_0)+\varepsilon_2\cdot(y-y_0) = \varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}, \qquad (7)$$

it is easy to see that (1) and (2) are equivalent. Their equivalence to (3) and (4) are obvious.

The following Theorem shows that the assumption of existence of $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ is not essential in the definition of differentiability.

**Theorem II**: If there exists $a,\ b,\ c \in \mathbb{R}$ such that the linear function

$$L(x, y) = a \cdot (x - x_0) + b \cdot (y - y_0) + c \tag{8}$$

satisfies

$$f(x_0, y_0) = L(x_0, y_0) \tag{9}$$

and

$$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \qquad \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0, \tag{2}$$

then $\partial_x f(x_0, y_0)$ and $\partial_y f(x_0, y_0)$ both exist and

$$a = \partial_x f(x_0, y_0),\ b = \partial_y f(x_0, y_0),\ c = f(x_0, y_0). \tag{10}$$

**Proof**:

It follows from (9) that $c = f(x_0, y_0)$.

Next, we compute $\partial_x f(x_0, y_0) = \lim_{x\to x_0} \dfrac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$ directly. From (2) and (8), we have

$$\lim_{x\to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{x\to x_0} \frac{L(x, y_0) + \varepsilon \cdot |x - x_0| - f(x_0, y_0)}{x - x_0}$$
$$= \lim_{x\to x_0} \frac{\big(f(x_0, y_0) + a \cdot (x - x_0)\big) + \varepsilon \cdot |x - x_0| - f(x_0, y_0)}{x - x_0} = \lim_{x\to x_0} (a \pm \varepsilon) = a.$$

The proof for $\partial_y f(x_0, y_0)$ is similar.

In view of Theorem II, we have a new and equivalent definition for differentiability:

**Definition 1'**:

> $f(x, y)$ is differentiable at $(x_0, y_0)$ if there exist $a,\ b,\ c \in \mathbb{R}$ such that the linear function
>
> $$L(x, y) = a \cdot (x - x_0) + b \cdot (y - y_0) + c \tag{8}$$
>
> satisfies
>
> $$f(x_0, y_0) = L(x_0, y_0) \tag{9}$$
>
> and
>
> $$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \qquad \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0. \tag{2}$$
>
> Here one can replace (2) by (1), (3), or (4).

**Remark 3**: It is straight forward to generalize Definition 1 and Definition 1' to higher dimensional case. For example, the 3D analogue of (2) reads

$$f(x, y, z) = L(x, y, z) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}, \lim_{(x,y,z)\to(x_0,y_0,z_0)} \varepsilon = 0, \tag{11}$$

where

$$L(x, y, z) = f(x_0, y_0, z_0) + \partial_x f(x_0, y_0, z_0)\cdot(x - x_0) + \partial_y f(x_0, y_0, z_0)\cdot(y - y_0) + \partial_z f(x_0, y_0, z_0)\cdot(z - z_0). \tag{12}$$

3D analogues of (1), (3) and (4) are similar.

**Question**: How do we verify whether $f(x,y)$ is differentiable at $(x_0,y_0)$? For example, when $f(x,y)$ is of the form

$$f(x,y) = \begin{cases} \cdots & (x,y) \neq (0.0) \\ 0 & (x,y) = (0.0) \end{cases}.$$

**Answer**:

**If you know the answer (differentiable or not)**:

**Theorems to prove $f$ is differentiable at $(x_0,y_0)$:**

Section 14.3, page 832, Theorem 3:

If $f_x$ and $f_y$ are continuous at $(x_0,y_0)$, then $f$ is differentiable at $(x_0,y_0)$.

Note: INCONCLUSIVE if $f_x$ or $f_y$ is NOT continuous at $(x_0,y_0)$.

**Theorems to prove $f$ is NOT differentiable at $(x_0,y_0)$:**

Section 14.3, page 832, Theorem 4:

If $f$ is not continuous at $(x_0,y_0)$, then it is not differentiable at $(x_0,y_0)$.

Note: INCONCLUSIVE if $f$ is continuous at $(x_0,y_0)$.

Section 14.5, page 847, Theorem 9:

If $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u},(x_0,y_0)} \neq \nabla f(x_0,y_0)\cdot\boldsymbol{u}$ for some direction $\boldsymbol{u}$, then $f$ is not differentiable at $(x_0,y_0)$.

**If you are not sure whether $f$ is differentiable at $(x_0,y_0)$ or not**:

**Step 1**: Find $\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$. If one of them does not exist, then $f(x,y)$ is NOT differentiable at $(x_0,y_0)$.

**Step 2**: If $\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$ exist, we still need to check whether $L(x,y)$ given by (5) satisfies (1), (2). (3) or (4). In general, it is easier to check (2). In other words, to check whether the following is true:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0\,? \tag{13}$$

If (13) is true, then we can write

$$\varepsilon \overset{\text{def}}{=} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, \qquad \lim_{(x,y)\to(x_0,y_0)} \varepsilon = 0. \tag{14}$$

which is the same as (2). Thus we conclude that $f(x,y)$ is differentiable at $(x_0,y_0)$ if and only if (13) holds. See Homework 09 problem 3 for some examples.

**Exercise 1** (Section 14.3) :
Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{15}$$

1. Show that

$$\partial_x f(x,y) = \begin{cases} \dfrac{y^3}{(\sqrt{x^2+y^2})^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \tag{16}$$

   Hint: For $(x,y) \neq (0,0)$, compute the derivative directly. For $(x,y) = (0,0)$, find the partial derivative from definition.

2. Use Two Path Test to show that $\partial_x f$ is not continuous at $(0,0)$.

   Since $\partial_x f$ is not continuous at $(0,0)$, Theorem 3 is not applicable and therefore the differentiability of $f$ at $(0,0)$ is still inconclusive. To find out whether $f$ is differentiable at $(0,0)$ or not, we continue with the following steps.

3. Find the linear function $L(x,y) = f(0,0) + \partial_x f(0,0)(x-0) + \partial_y f(0,0)(y-0)$.

4. Check if this $L(x,y)$ satisfies any of the equivalent definition of differentiability (1)-(4). In this example, version (2) (i.e. $\varepsilon \cdot \sqrt{(x-0)^2+(y-0)^2}$) is easier to check. In the end, you should reach the conclusion that $f$ is not differentiable at $(0,0)$.

**Exercise 2** (Section 14.3-14.5) : Are following True or False? Find the corresponding Theorem if true, a counter example if false.

1. $f(x,y)$ is differentiable at $(x_0,y_0)$ $\overset{?}{\Longrightarrow}\!\!\!\!\overset{?}{\Longleftarrow}$ $f(x,y)$ is continuous at $(x_0,y_0)$.

2. $f(x,y)$ is differentiable at $(x_0,y_0)$ $\overset{?}{\Longrightarrow}\!\!\!\!\overset{?}{\Longleftarrow}$ $\partial_x f(x_0,y_0)$ and $\partial_y f(x_0,y_0)$ both exist.

3. (Hard!)

   $f(x,y)$ is differentiable at $(x_0,y_0)$ $\overset{?}{\Longrightarrow}\!\!\!\!\overset{?}{\Longleftarrow}$ $\partial_x f(x,y)$ and $\partial_y f(x,y)$ both are continuous at $(x_0,y_0)$.

4. $f(x,y)$ is differentiable at $(x_0,y_0)$ $\overset{?}{\Longrightarrow}\!\!\!\!\overset{?}{\Longleftarrow}$ The directional derivative $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u},\,(x_0,y_0)}$ exists for any unit vector $\boldsymbol{u}$.

5. (Harder! Think about it only if time permits.)

   $f(x,y)$ is differentiable at $(x_0,y_0)$ $\overset{?}{\Longrightarrow}\!\!\!\!\overset{?}{\Longleftarrow}$ The directional derivative $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u},\,(x_0,y_0)}$ exists for any unit vector $\boldsymbol{u}$ and satisfies $\left(\dfrac{df}{ds}\right)_{\boldsymbol{u},(x_0,y_0)} = \nabla f(x_0,y_0)\cdot\boldsymbol{u}$.