

## Remark on Definition of Differentiability (v03)

The following definitions are equivalent (i.e. the same, even though they look different)

**Definition 1:**  $f(x, y)$  is differentiable at  $(x_0, y_0)$  if

$\partial_x f(x_0, y_0)$  and  $\partial_y f(x_0, y_0)$  exist and

$$f(x, y) = L(x, y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0), \quad \lim_{(x,y) \rightarrow (x_0,y_0)} \varepsilon_1 = \lim_{(x,y) \rightarrow (x_0,y_0)} \varepsilon_2 = 0, \quad (1)$$

(other equivalent definitions include:

$$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} \varepsilon = 0, \quad (2)$$

or

$$\Delta z = \partial_x f(x_0, y_0) \Delta x + \partial_y f(x_0, y_0) \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y \quad (\text{textbook version}) \quad (3)$$

or

$$\Delta z = \partial_x f(x_0, y_0) \Delta x + \partial_y f(x_0, y_0) \Delta y + \varepsilon \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad (4)$$

where

$$L(x, y) = f(x_0, y_0) + \partial_x f(x_0, y_0) \cdot (x - x_0) + \partial_y f(x_0, y_0) \cdot (y - y_0), \quad (5)$$

and

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta z = f(x, y) - f(x_0, y_0). \quad (6)$$

**Remark 1:**

In view of the identity (see homework 07, problem 2 for hint of proof)

$$\varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0) = \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad (7)$$

it is easy to see that (1) and (2) are equivalent. Their equivalence to (3) and (4) are obvious.

The following Theorem shows that the assumption of existence of  $\partial_x f(x_0, y_0)$  and  $\partial_y f(x_0, y_0)$  is not essential in the definition of differentiability.

**Theorem II:** If there exists  $a, b, c \in \mathbb{R}$  such that the linear function

$$L(x, y) = a \cdot (x - x_0) + b \cdot (y - y_0) + c \quad (8)$$

satisfies

$$f(x_0, y_0) = L(x_0, y_0) \quad (9)$$

and

$$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} \varepsilon = 0, \quad (2)$$

then  $\partial_x f(x_0, y_0)$  and  $\partial_y f(x_0, y_0)$  both exist and

$$a = \partial_x f(x_0, y_0), \quad b = \partial_y f(x_0, y_0), \quad c = f(x_0, y_0). \quad (10)$$

**Proof:**

It follows from (9) that  $c = f(x_0, y_0)$ .

Next, we compute  $\partial_x f(x_0, y_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$  directly. From (2) and (8), we have

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{L(x, y_0) + \varepsilon \cdot |x - x_0| - f(x_0, y_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{(f(x_0, y_0) + a \cdot (x - x_0)) + \varepsilon \cdot |x - x_0| - f(x_0, y_0)}{x - x_0} = \lim_{x \rightarrow x_0} (a \pm \varepsilon) = a. \end{aligned}$$

The proof for  $\partial_y f(x_0, y_0)$  is similar.

In view of Theorem II, we have a new and equivalent definition for differentiability:

**Definition 1':**

$f(x, y)$  is differentiable at  $(x_0, y_0)$  if there exist  $a, b, c \in \mathbb{R}$  such that the linear function

$$L(x, y) = a \cdot (x - x_0) + b \cdot (y - y_0) + c \quad (8)$$

satisfies

$$f(x_0, y_0) = L(x_0, y_0) \quad (9)$$

and

$$f(x, y) = L(x, y) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} \varepsilon = 0. \quad (2)$$

Here one can replace (2) by (1), (3), or (4).

**Remark 3:** It is straight forward to generalize Definition 1 and Definition 1' to higher dimensional case. For example, the 3D analogue of (2) reads

$$f(x, y, z) = L(x, y, z) + \varepsilon \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}, \quad \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} \varepsilon = 0, \quad (11)$$

where

$$L(x, y, z) = f(x_0, y_0, z_0) + \partial_x f(x_0, y_0, z_0) \cdot (x - x_0) + \partial_y f(x_0, y_0, z_0) \cdot (y - y_0) + \partial_z f(x_0, y_0, z_0) \cdot (z - z_0). \quad (12)$$

3D analogues of (1), (3) and (4) are similar.

**Question:** How do we verify whether  $f(x, y)$  is differentiable at  $(x_0, y_0)$ ? For example, when  $f(x, y)$  is of the form

$$f(x, y) = \begin{cases} \cdots & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

**Answer:**

**If you know the answer (differentiable or not):**

**Theorems to prove  $f$  is differentiable at  $(x_0, y_0)$ :**

Section 14.3, page 832, Theorem 3:

If  $f_x$  and  $f_y$  are continuous at  $(x_0, y_0)$ , then  $f$  is differentiable at  $(x_0, y_0)$ .

Note: INCONCLUSIVE if  $f_x$  or  $f_y$  is NOT continuous at  $(x_0, y_0)$ .

**Theorems to prove  $f$  is NOT differentiable at  $(x_0, y_0)$ :**

Section 14.3, page 832, Theorem 4:

If  $f$  is not continuous at  $(x_0, y_0)$ , then it is not differentiable at  $(x_0, y_0)$ .

Note: INCONCLUSIVE if  $f$  is continuous at  $(x_0, y_0)$ .

Section 14.5, page 847, Theorem 9:

If  $\left(\frac{df}{ds}\right)_{\mathbf{u}, (x_0, y_0)} \neq \nabla f(x_0, y_0) \cdot \mathbf{u}$  for some direction  $\mathbf{u}$ , then  $f$  is not differentiable at  $(x_0, y_0)$ .

**If you are not sure whether  $f$  is differentiable at  $(x_0, y_0)$  or not:**

**Step 1:** Find  $\partial_x f(x_0, y_0)$  and  $\partial_y f(x_0, y_0)$ . If one of them does not exist, then  $f(x, y)$  is NOT differentiable at  $(x_0, y_0)$ .

**Step 2:** If  $\partial_x f(x_0, y_0)$  and  $\partial_y f(x_0, y_0)$  exist, we still need to check whether  $L(x, y)$  given by (5) satisfies (1), (2), (3) or (4). In general, it is easier to check (2). In other words, to check whether the following is true:

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0? \quad (13)$$

If (13) is true, then we can write

$$\varepsilon \stackrel{\text{def}}{=} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}, \quad \lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon = 0. \quad (14)$$

which is the same as (2). Thus we conclude that  $f(x, y)$  is differentiable at  $(x_0, y_0)$  if and only if (13) holds. See Homework 09 problem 3 for some examples.

**Exercise** (Section 14.3) :  
Consider the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \quad (15)$$

1. Show that

$$\partial_x f(x, y) = \begin{cases} \frac{y^3}{(\sqrt{x^2 + y^2})^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \quad (16)$$

Hint: For  $(x, y) \neq (0, 0)$ , compute the derivative directly. For  $(x, y) = (0, 0)$ , find the partial derivative from definition.

2. Use Two Path Test to show that  $\partial_x f$  is not continuous at  $(0, 0)$ .

Since  $\partial_x f$  is not continuous at  $(0, 0)$ , Theorem 3 is not applicable and therefore the differentiability of  $f$  at  $(0, 0)$  is still inconclusive. To find out whether  $f$  is differentiable at  $(0, 0)$  or not, we continue with the following steps.

3. Find the linear function  $L(x, y) = f(0, 0) + \partial_x f(0, 0)(x - 0) + \partial_y f(0, 0)(y - 0)$ .

4. Check if this  $L(x, y)$  satisfies any of the equivalent definition of differentiability (1)-(4). In this example, version (2) (i.e.  $\varepsilon \cdot \sqrt{(x - 0)^2 + (y - 0)^2}$ ) is easier to check. In the end, you should reach the conclusion that  $f$  is not differentiable at  $(0, 0)$ .