

Green's Theorem

C : a simple closed curve

$\vec{F} = (M, N)$, M, N and

their first derivatives are cont.

in $R = \text{interior of } C$

Then

$$(i) \oint_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx = \iint_R (M_x + N_y) dA$$

(normal form) $\vec{n} ds = (dy, -dx) = \left(\frac{dy}{dt}, -\frac{dx}{dt} \right) dt$

$$(ii) \oint_C \vec{F} \cdot \vec{T} ds = \oint_C M dx + N dy = \iint_R (N_x - M_y) dA$$

(tangential form)

$$\vec{T} ds = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) dt$$

Rm $\vec{F} = (M, N) \leftrightarrow \vec{G} = (N, -M)$

normal form for \vec{F} = tangential form for \vec{G}
(tangential) (normal)

Rm They are special cases of

(Gauss) $\iiint_D \nabla \cdot \vec{F} \, dV = \iint_{\partial D} \vec{F} \cdot \vec{n} \, d\sigma$
(div \vec{F})

(Stokes) $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T} \, ds$

$d\sigma$: surface integral

∂D : boundary of D (domain)

∂S : boundary of S (surface)

dt

where

$$\begin{aligned}\operatorname{div} \vec{F} &= \nabla \cdot \vec{F} = (\partial_x, \partial_y, \partial_z) \cdot (F_1, F_2, F_3) \\ &= \partial_x F_1 + \partial_y F_2 + \partial_z F_3\end{aligned}$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = (\partial_x, \partial_y, \partial_z) \times (F_1, F_2, F_3)$$

$$\begin{aligned}&= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1)\end{aligned}$$

Special case of Stokes': Take $S = R$. $\partial S = C$

$$\vec{F} = (F_1, F_2, 0) = (M, N, 0), \quad \vec{n} = (0, 0, 1) \Leftrightarrow C = \odot$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \iint_R \underbrace{\partial_x F_2 - \partial_y F_1}_{N_x - M_y} \, dA = \oint_C \vec{F} \cdot \vec{T} \, ds$$

Eg 3 Verify both forms
of Green's Thm for

$$\vec{F}(x, y) = (x - y, x) \text{ and}$$

$$C: \vec{r}(t) = (\cos t, \sin t), \quad 0 \leq t \leq 2\pi$$

Sol normal form

$$\iint_{x^2 + y^2 \leq 1} M_x + N_y \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$$\text{LHS} = \iint_{x^2 + y^2 \leq 1} 1 \, dA = \pi$$

$$\text{RHS} = \oint_C \vec{F} \cdot \left(\frac{dy}{dt}, -\frac{dx}{dt} \right) dt = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} dt = \pi$$

Tangential form

$$\iint_{x^2+y^2 \leq 1} (N_x - M_y) dA = \oint_C M dx + N dy$$

$$\text{LHS} = \iint_{x^2+y^2 \leq 1} 2 dA = 2\pi$$

$$\begin{aligned} \text{RHS} &= \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt \\ &= \int_0^{2\pi} -\cos t \sin t dt + \int_0^{2\pi} 1 dt \\ &= 2\pi \end{aligned}$$

Ex 4 Evaluate $\oint_C \vec{F} \cdot \vec{T} ds$

where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (M, N)$

$$C: x^2 + y^2 = a^2$$

Sol.: Component test

$$M_y - N_x = 0$$

Method 1: Green's Thm (tangential)

$$\iint_{x^2+y^2 \leq a^2} 0 \, dA = 0 \quad (\text{Wrong!})$$

Method 2: $x = a \cos t, y = a \sin t$

$$\text{Ans} = \int_0^{2\pi} \begin{pmatrix} -\frac{\sin t}{a} \\ \frac{\cos t}{a} \end{pmatrix} \cdot \begin{pmatrix} -a \sin t \\ a \cos t \end{pmatrix} dt = 2\pi \quad (\text{correct})$$

Note: M, N and their first derivatives are not cont. in $x^2 + y^2 \leq a^2$!

Ex 4. Evaluate $\oint_C \vec{F} \cdot \vec{T} ds$

where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ and

$$C: \frac{x^2}{4} + y^2 = 1$$

Sol



$$(A): \stackrel{\text{Green's}}{=} \iint (M_y - N_x) dA = 0 \quad \text{Ans: } = 2\pi$$

$$(B) \stackrel{\text{I.I.V.}}{=} \oint_{x^2+y^2=a^2, 0 < a < 1} M dx + N dy = 2\pi$$

Pf of Green's Theorem (tangential form)

$$(i) \oint_C \vec{F} \cdot \vec{T} ds = \oint_C M dx + N dy = \iint_R (N_x - M_y) dA$$

(Pf for normal form is similar)



If (i) is true for (C_1, R_1) , (C_2, R_2) then it is true for (C, R) where



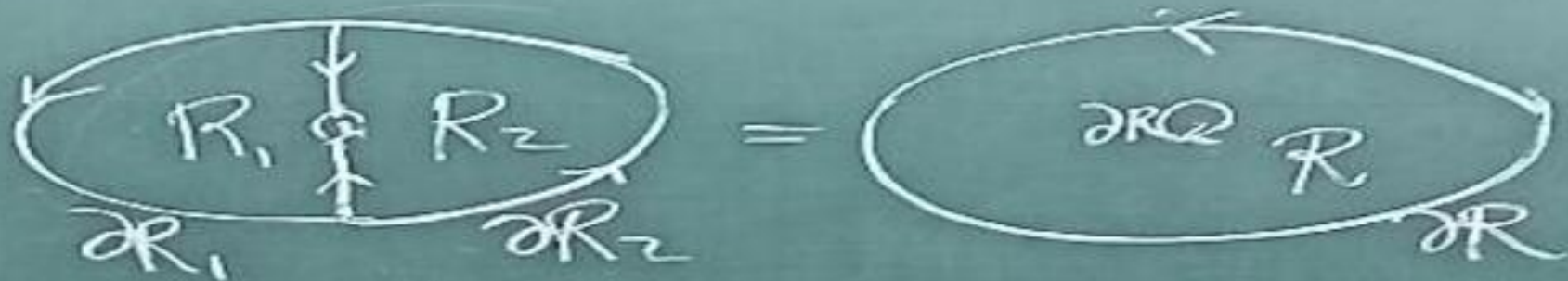
$$C = C_1 + C_2$$

$$R = R_1 \cup R_2$$

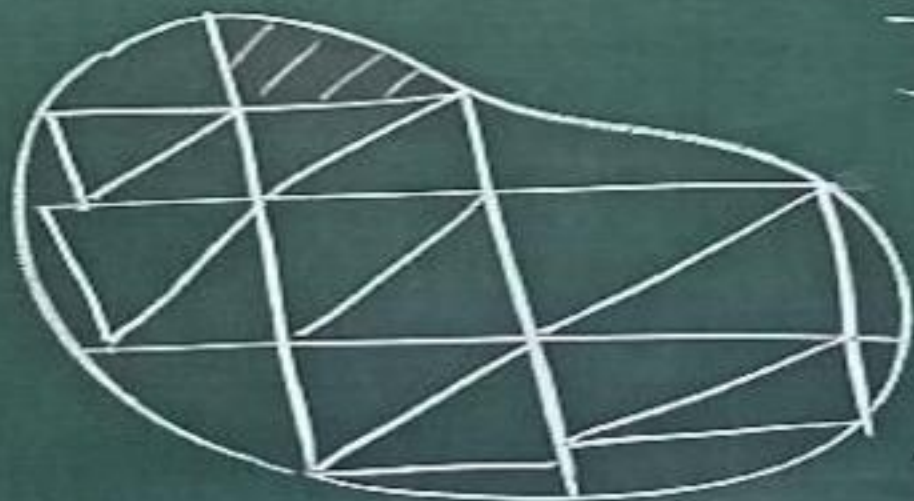
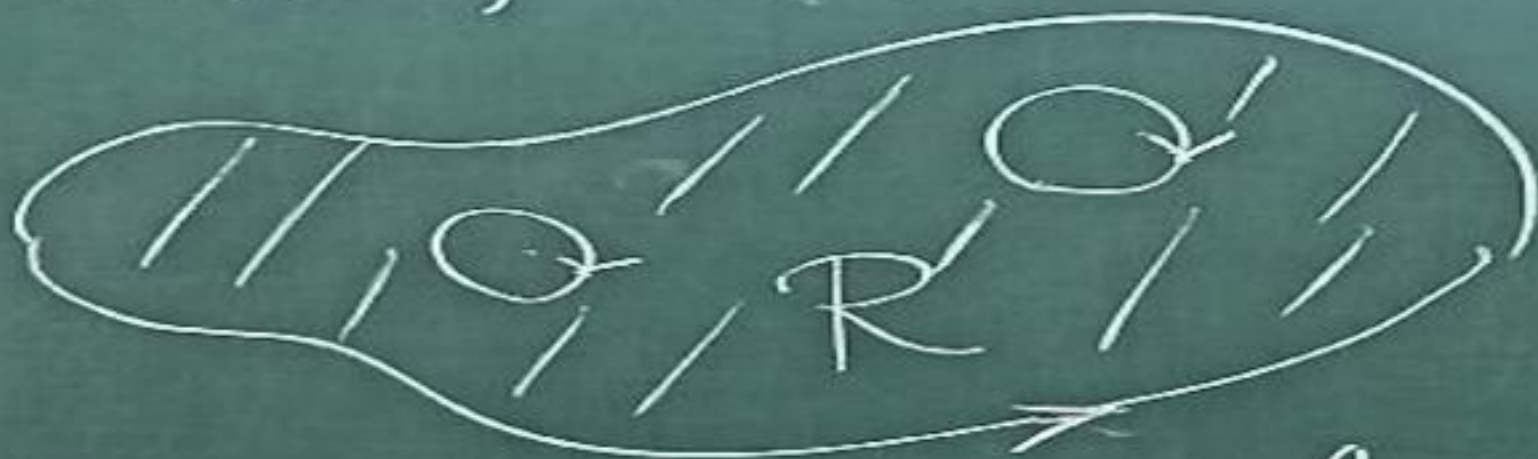
$$C = C_1 + C_2$$


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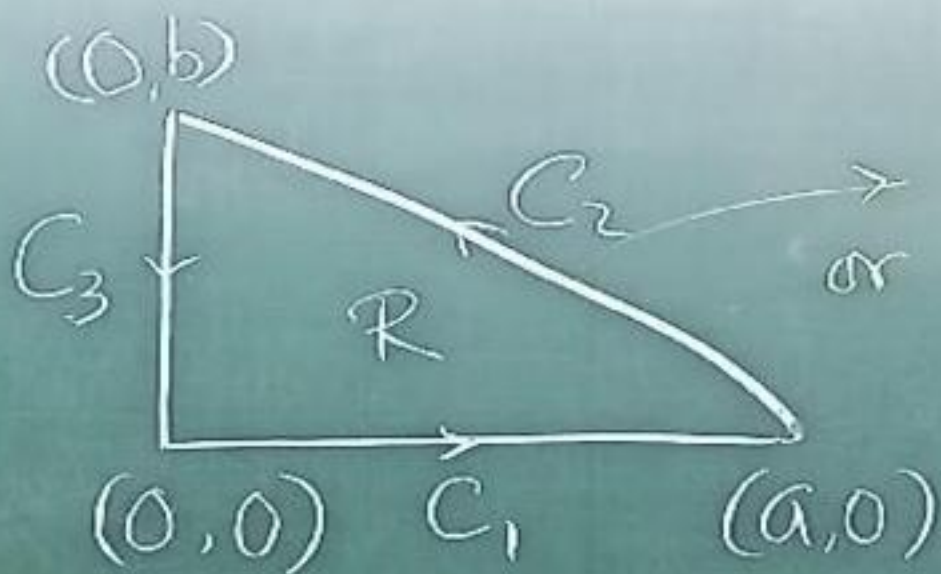
Rm in previous example



Similarly



In general,
it suffice to
verify R
of the form 



$$y = f(x) \quad 0 \leq x \leq a$$

$$\text{or} \quad x = g(y) \quad 0 \leq y \leq b$$

$$C_2 = \{(x, f(x)), 0 \leq x \leq a\}$$

$$= \{(g(y), y), 0 \leq y \leq b\}$$

Check tangential form, line integrals:

$$\int_{C_1} \vec{F} \cdot \vec{T} \, dt = \int_0^a \vec{F} \cdot (1, 0) \, dt = \int_0^a M(t, 0) \, dt = \int_0^a M(x, 0) \, dx \quad (\text{I})$$

$$\int_{C_3} \vec{F} \cdot \vec{T} \, dt = \int_0^b \vec{F} \cdot (0, -1) \, dt = - \int_{y=b}^0 N(0, y) \, dy$$

$$\text{or} \quad \int_{C_3 \downarrow} \vec{F} \cdot \vec{T} \, ds + \int_{C_3 \uparrow} \vec{F} \cdot \vec{T} \, ds = - \int_{s=0}^b N(0, s) \, ds = - \int_{y=0}^b N(0, y) \, dy \quad (\text{III})$$

$$\int_{C_2} \left(M \frac{dx}{dt} + N \frac{dy}{dt} \right) dt = \int_0^b \left(M(g(t), t) g'(t) + N(g(t), t) \right) dt$$

$$C_2 = \{y=t, x=g(t), 0 \leq t \leq b\} \quad \dot{x} = g'(t), \dot{y} = 1$$

Double integrals $\iint_R (N_x - M_y) dA$

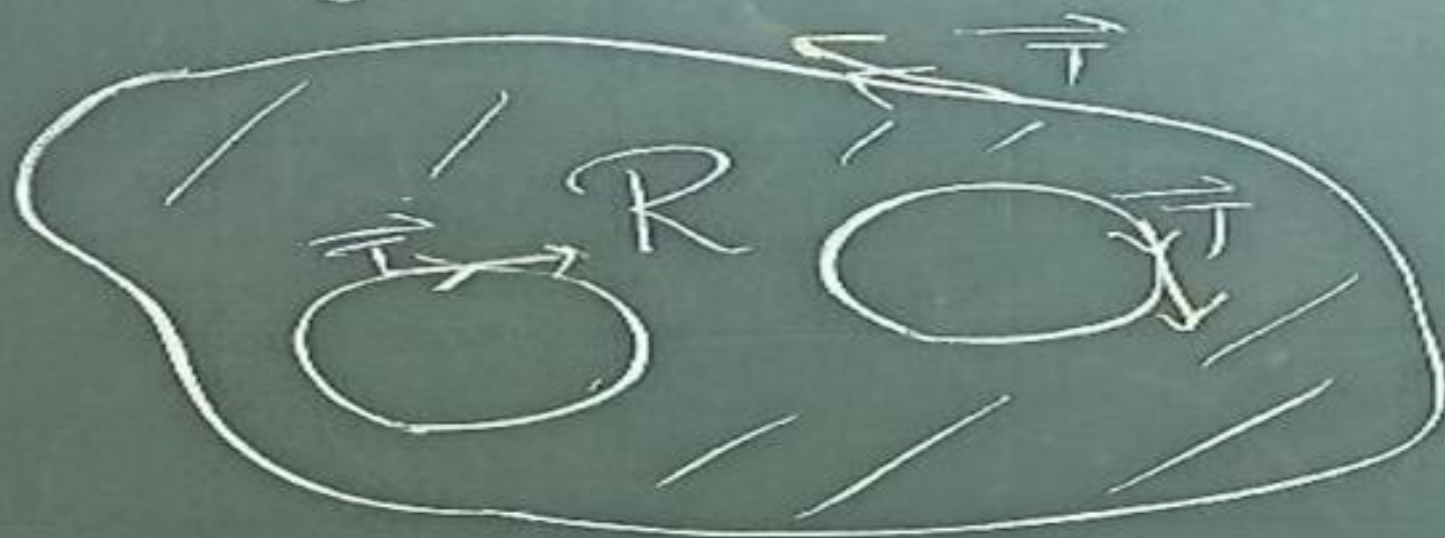
$$\iint_R N_x dA = \int_0^b \int_0^{g(y)} N_x dx dy = \int_0^b (N(g(y), y) - N(0, y)) dy$$

$$- \iint_R M_y dA = - \int_0^a \int_{y=0}^{f(x)} M_y dy dx = - \int_0^a (M(x, f(x)) - M(x, 0)) dx$$

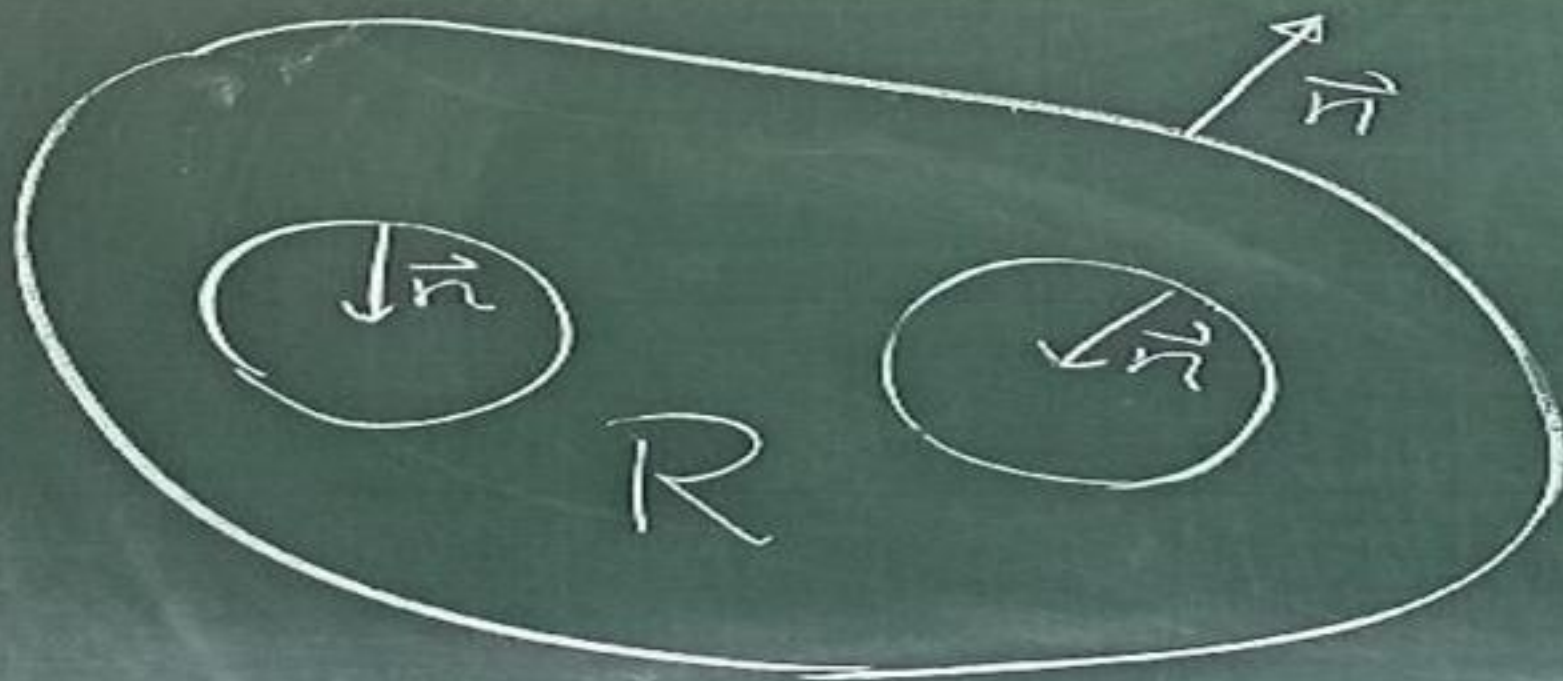
$$(IV) = - \int_0^a M(x, f(x)) dx \quad y = f(x) \\ x = f^{-1}(y) \equiv g(y)$$

$$= - \int_{y=b}^0 M(g(y), y) g'(y) dy \quad dx = g'(y) dy \quad \therefore (i) \text{ verified}$$

Tangential form



Normal form



RM For $\vec{F} = \frac{(-y, x)}{x^2 + y^2}$

$$\oint_C \vec{F} \cdot \vec{T} ds = \begin{cases} 2\pi, & (0,0) \in R \\ 0, & (0,0) \notin R \end{cases}$$

$R =$ inside of C

RM $\frac{(-y, x)}{x^2 + y^2} = \nabla \tan^{-1}\left(\frac{y}{x}\right)$

But $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is NOT everywhere defined on $x^2 + y^2 \neq 0$