Green's Theorem

C: a simple closed curve

$\vec{F} = (M, N)$, M, N and their first derivatives are cont. in R = interior of C

Then

(i) $\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\, dA$

(normal form) $\vec{n}\, ds = (dy, -dx) = \left(\frac{dy}{dt}, \frac{-dx}{dt}\right) dt$

(ii) $\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$

(tangential form) $\vec{T}\, ds = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt$

Rm $\vec{F} = (M, N) \longleftrightarrow \vec{G} = (N, -M)$

normal form for $\vec{F}$ = tangential form for $\vec{G}$
(tangential) (normal)

Rm They are special cases of

(Gauss) $\iiint_D \nabla \cdot \vec{F} \, dV = \iint_{\partial D} \vec{F} \cdot \vec{n} \, d\sigma$
$(\operatorname{div} \vec{F})$

(Stokes') $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T} \, ds$

$d\sigma$: surface integral

∂D: boundary of D (domain)

∂S: boundary of S (surface)

dt

whee

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} \ \left(= (\partial_x, \partial_y, \partial_z)\cdot(F_1, F_2, F_3)\right)$$

$$= \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} \ \left(= (\partial_x, \partial_y, \partial_z)\times(F_1, F_2, F_3)\right)$$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\partial_y F_3 - \partial_z F_2,\ \partial_z F_1 - \partial_x F_3,\ \partial_x F_2 - \partial_y F_1\right)$$

Special case of Stokes': Take $S = R$. $\partial S = C$

$\vec{F} = (\overset{F_1}{M}, \overset{F_2}{N}, 0)$, $\vec{n} = (0,0,1) \Leftrightarrow C = \circlearrowleft$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma = \iint_R \underbrace{\partial_x F_2 - \partial_y F_1}_{\overset{\shortparallel}{N_x - M_y}} dA = \oint_C \vec{F}\cdot\vec{T}\, ds$$

Eg3 Verify both forms of Green's Thm for

$\vec{F}(x,y) = (x-y, x)$ and

$C: \vec{r}(t) = (\cos t, \sin t),\ 0 \le t \le 2\pi$

Sol normal form

$$\iint_{x^2+y^2 \le 1} M_x + N_y \, dA = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$$LHS = \iint_{x^2+y^2 \le 1} 1 \, dA = \pi$$

$$RHS = \oint_C \vec{F} \cdot \left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} dt = \pi$$

Tangential form

$$\iint_{x^2+y^2\le 1} (N_x - M_y)\,dA = \oint_C M\,dx + N\,dy$$

$$\text{LHS} = \iint_{x^2+y^2\le 1} 2\,dA = 2\pi$$

$$\text{RHS} = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} -\cos t \sin t \, dt + \int_0^{2\pi} 1 \, dt$$

$$= 2\pi.$$

Eg4 Evaluate $\oint_C \vec{F} \cdot \vec{T}\, ds$

where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (M, N)$

$C: x^2 + y^2 = a^2$

Sol: Component test

$M_y - N_x = 0$

Method 1: Green's Thm (tangential)

$\iint_{x^2+y^2 \le a^2} 0 \, dA = 0$ (Wrong!)

Method 2: $x = a\cos t$, $y = a\sin t$

$\text{Ans} = \int_0^{2\pi} \begin{pmatrix} \frac{-\sin t}{a} \\ \frac{\cos t}{a} \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix} dt = 2\pi$ (correct)

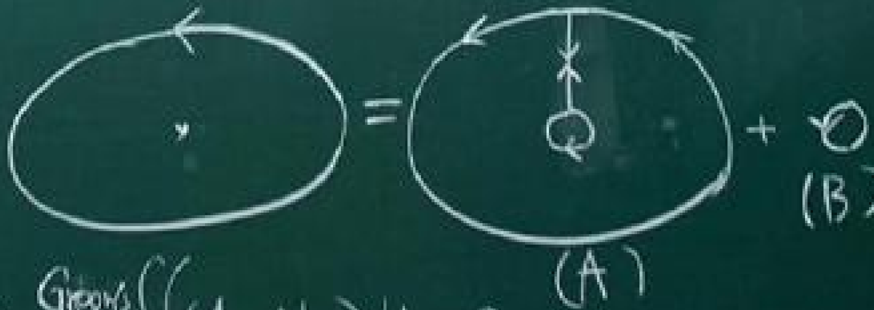
Note: M, N and their first derivatives are not cont. in $x^2+y^2 \le a^2$!

Eg4. Evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$

where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ and

$C: \frac{x^2}{4}+y^2 = 1$

Sol



$(A): \overset{\text{Green's}}{=} \iint (M_y - N_x)\,dA = 0$

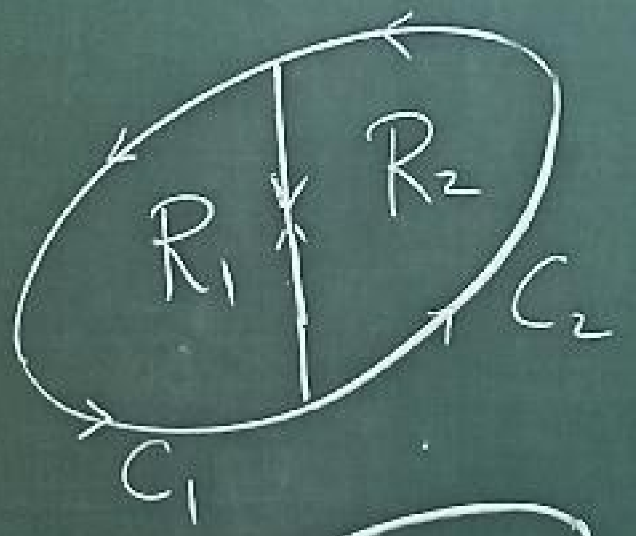
Ans: $= 2\pi$

$(B) \overset{\text{HW}}{=} \oint_{x^2+y^2=a^2,\ 0<a<1} M\,dx + N\,dy = 2\pi$

Pf of Green's Theorem (tangential form)

(i) $\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$

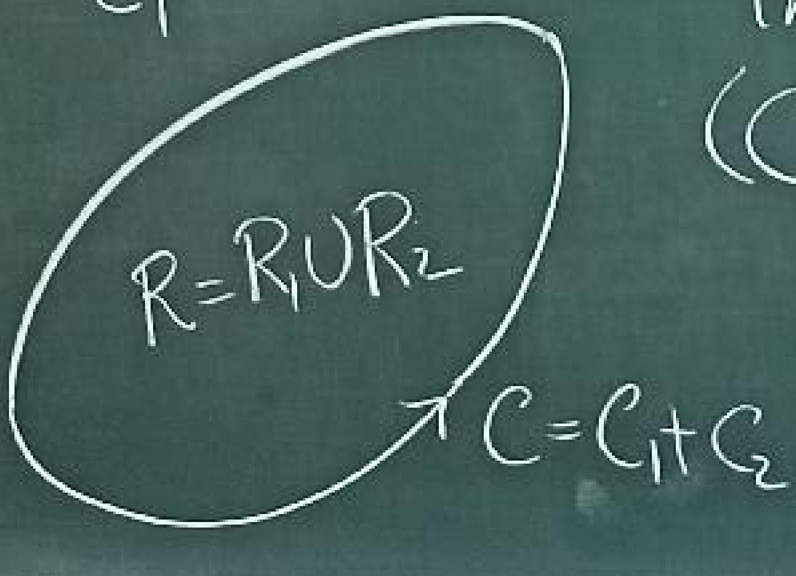
(pf for normal form is similar)



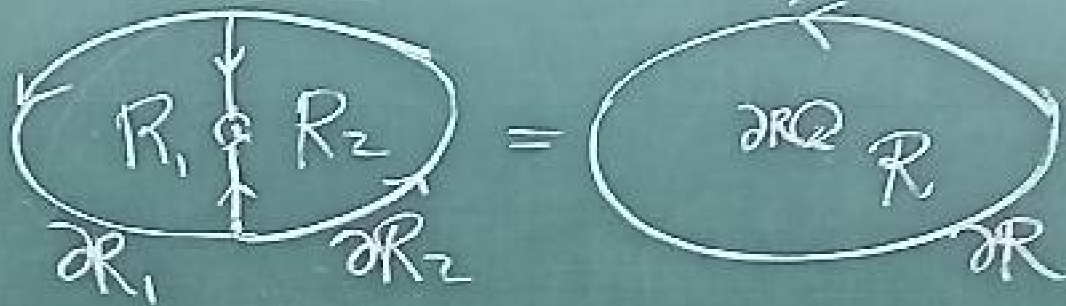
If (i) is true for (C_1,R_1), (C_2,R_2) then it is true for (C,R) where

$C = C_1 + C_2$

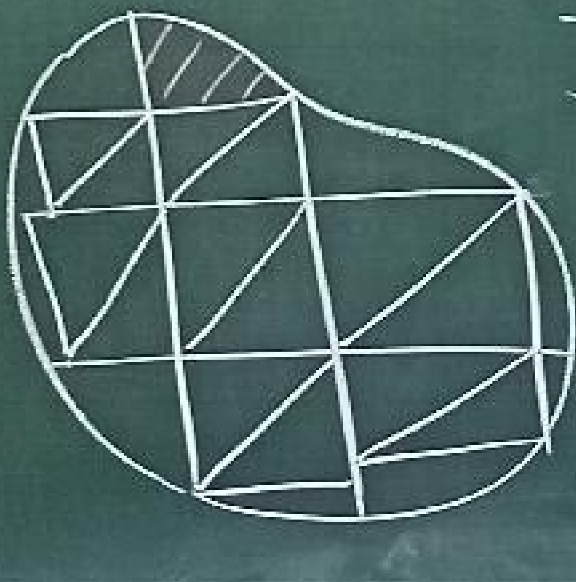
$R = R_1 \cup R_2$



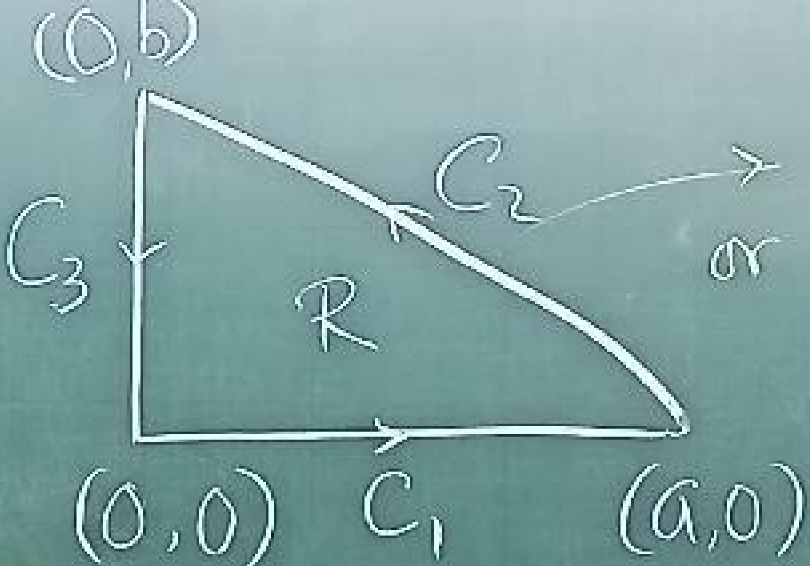
Rm in previous example



Similarly



In general,
it suffice to
verify R
of the form 



$y = f(x) \quad 0 \le x \le a$

or $x = g(y) \quad 0 \le y \le b$

$C_2 = \{(x, f(x)),\ 0 \le x \le a\}$
$= \{(g(y), y),\ 0 \le y \le b\}$

Check tangential form, line integrals:

$$\int_{C_1} \underset{\vec{T}=(1,0)}{\overset{t=x}{=}} \int_{t=0}^{a} \vec{F}\cdot(1,0)\,dt = \int_0^a M(t,0)\,dt = \int_0^a M(x,0)\,dx \quad (\mathrm{I})$$

$$\int_{C_3} \underset{\vec{T}=(0,-1)}{\overset{t=b-y}{=}} \int_{t=0}^{b} -N(0,b-t)\,dt = -\int_{y=b}^{0} N(0,y)(-dy)$$

$$\text{or } \int_{C_3\downarrow} = -\int_{-C_3\uparrow} \underset{\vec{T}=(0,1)}{\overset{s=y}{=}} -\int_{s=0}^{b} N(0,s)\,ds = -\int_{y=0}^{b} N(0,y)\,dy \quad (\mathrm{III})$$

$$\int_{C_2} \underset{t=y}{=} \int_{t=0}^{b} \left(M\frac{dx}{dt} + N\frac{dy}{dt} \right) dt = \int_0^b \Big(\underset{(\mathrm{II}_1)}{M(g(t),t)\,g'(t)} + \underset{(\mathrm{II}_2)}{N(g(t),t)} \Big) dt$$

$$C_2 = \{ y=t,\ x=g(t),\ 0 \le t \le b \} \quad \dot{x} = g'(t),\ \dot{y} = 1$$

Double integrals $\iint_R (N_x - M_y)\, dA$

$$\iint_R N_x\, dA = \int_0^b \int_0^{g(y)} N_x\, dx\, dy = \int_0^b \big(\underset{(\mathrm{II}_2)}{N(g(y),y)} - \underset{(\mathrm{III})}{N(0,y)} \big) dy$$

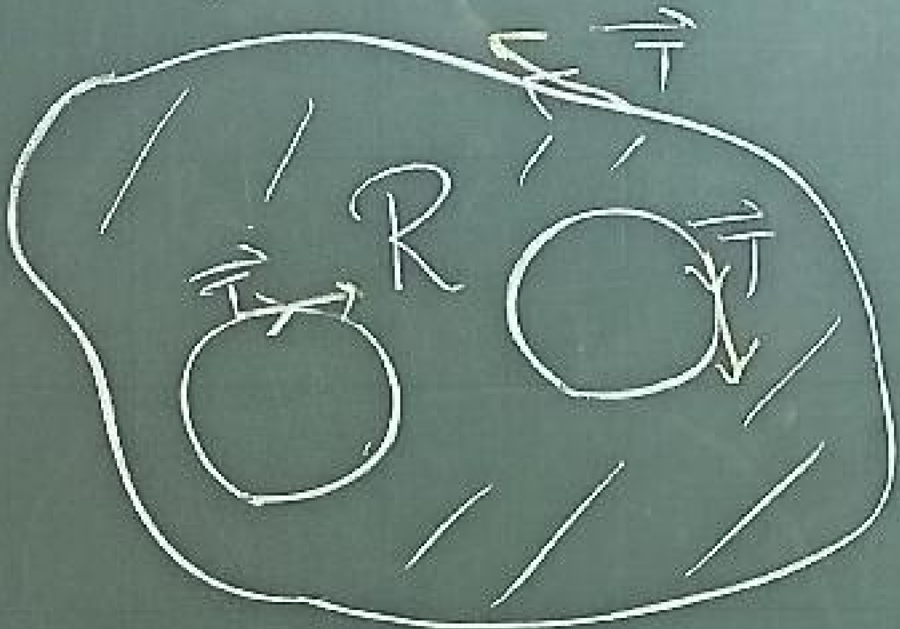
$$-\iint_R M_y\, dA = -\int_0^a \int_{y=0}^{f(x)} M_y\, dy\, dx = -\int_0^a \big(\underset{(\mathrm{IV})}{M(x,f(x))} - \underset{(\mathrm{I})}{M(x,0)} \big) dx$$

$$(\mathrm{IV}) = -\int_0^a M(x,f(x))\, dx \qquad \begin{array}{l} y = f(x) \\ x = f^{-1}(y) \equiv g(y) \end{array}$$

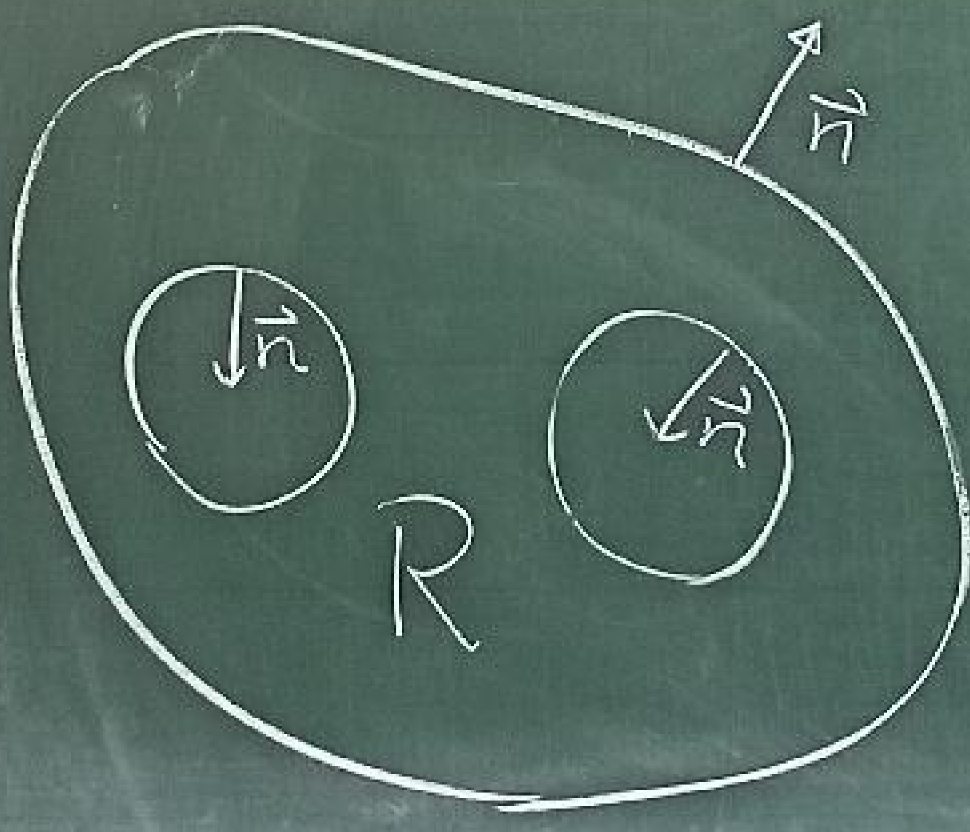
$$= -\int_{y=b}^{0} M(g(y),y)\, g'(y)\, dy \qquad dx = g'(y)\, dy \qquad \therefore \text{(i) verified}$$

$$= (\mathrm{II}_1)$$

Tangential form



Normal form



$\underline{\underline{\text{Rm}}}$ For $\vec{F} = \dfrac{(-y, x)}{x^2+y^2}$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \begin{cases} 2\pi, & (0,0) \in R \\ 0, & (0,0) \notin R \end{cases}$$

R = inside of C

[$\underline{\underline{\text{Rm}}}$ $\dfrac{(-y, x)}{x^2+y^2} = \nabla \tan^{-1}\left(\dfrac{y}{x}\right)$

But $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ is NOT everywhere defined on $x^2+y^2 \neq 0$]