Green's Theorem

C : a Simple closed curve

$\vec{F} = (M, N)$, M, N and

their first derivatives are cont.

in R = interier of C

Then

(i) $\displaystyle\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\,dA$

(normal form) $\vec{n}\,ds = (dy, -dx) = \left(\frac{dy}{dt}, \frac{-dx}{dt}\right)dt$

(ii) $\displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$

(tangential form) $\vec{T}\,ds = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)dt$

Rm $\vec{F} = (M, N) \longleftrightarrow \vec{G} = (N, -M)$

normal form for $\vec{F}$ = tangential form for $\vec{G}$
(tangential) (normal)

Rm They are special cases of

(Gauss) $\iiint_D \nabla \cdot \vec{F} \, dV = \iint_{\partial D} \vec{F} \cdot \vec{n} \, d\sigma$
$(\operatorname{div} \vec{F})$

(Stokes') $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T} \, ds$

$d\sigma$: surface integral
∂D: boundary of D (domain)
∂S: boundary of S (surface)

)dt

where

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} \; \left(= (\partial_x, \partial_y, \partial_z)\cdot(F_1, F_2, F_3)\right)$$
$$= \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} \; \left(= (\partial_x, \partial_y, \partial_z)\times(F_1, F_2, F_3)\right)$$
$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left(\partial_y F_3 - \partial_z F_2,\ \partial_z F_1 - \partial_x F_3,\ \partial_x F_2 - \partial_y F_1\right)$$

Special case of Stokes': Take $S = R$. $\partial S = C$

$$\vec{F} = (\overset{F_1}{M}, \overset{F_2}{N}, 0), \quad \vec{n} = (0,0,1) \Leftrightarrow C = \circlearrowleft$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \iint_R \underbrace{\partial_x F_2 - \partial_y F_1}_{= N_x - M_y}\, dA = \oint_C \vec{F}\cdot\vec{T}\, ds$$

Eg 3 Verify both forms of Green's Thm for

$\vec{F}(x,y) = (x-y, x)$ and

$C: \vec{r}(t) = (\cos t, \sin t), \ 0 \le t \le 2\pi$

<u>Sol</u> normal form

$$\iint_{x^2+y^2\le 1} M_x + N_y \, dA = \oint_C \vec{F}\cdot\vec{n}\, ds$$

$$\text{LHS} = \iint_{x^2+y^2\le 1} 1 \, dA = \pi$$

$$\text{RHS} = \oint_C \vec{F}\cdot\left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} dt = \pi$$

Tangential form

$$\iint_{x^2+y^2\le 1} (N_x - M_y)\,dA = \oint_C M\,dx + N\,dy$$

$$\text{LHS} = \iint_{x^2+y^2\le 1} 2\,dA = 2\pi$$

$$\text{RHS} = \int_0^{2\pi} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} -\cos t \sin t\,dt + \int_0^{2\pi} 1\,dt$$

$$= 2\pi.$$

Eg 4 Evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$

where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = (M, N)$

$C: x^2+y^2 = a^2$

Sol: Component test

$M_y - N_x = 0$

Method 1: Green's Thm (tangential)

$\iint_{x^2+y^2\le a^2} 0\,dA = 0$ (Wrong!)

Method 2: $x = a\cos t,\ y = a\sin t$

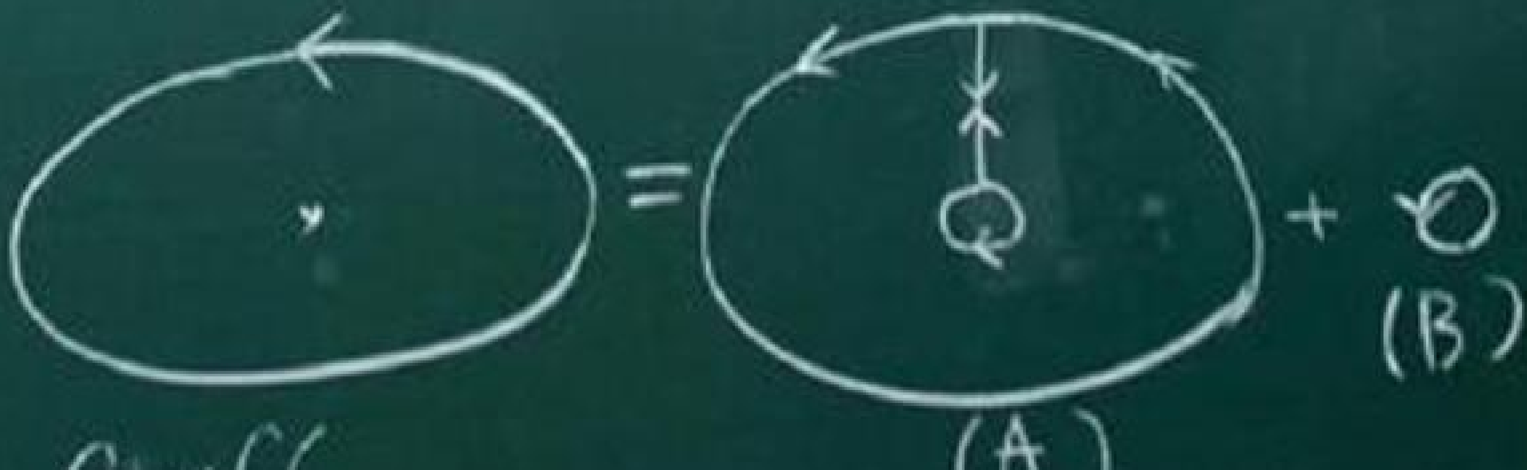
$\text{Ans} = \int_0^{2\pi} \begin{pmatrix} \frac{-\sin t}{a} \\ \frac{\cos t}{a} \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix} dt = 2\pi$ (correct)

Note: M, N and their first derivatives are not cont. in $x^2+y^2 \le a^2$!

Eg4. Evaluate $\oint_C \vec{F}\cdot\vec{T}\,ds$ where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ and

$C: \frac{x^2}{4}+y^2 = 1$

Sol



(A): $\overset{\text{Green's}}{=} \iint (M_y - N_x)\,dA = 0$

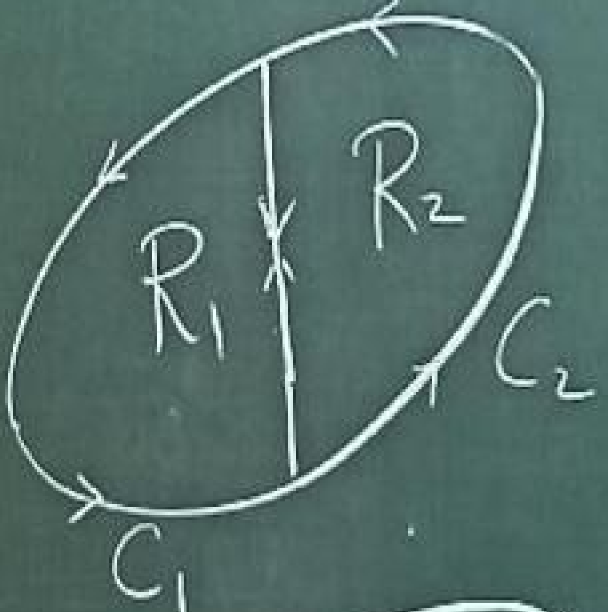
Ans: $= 2\pi$

(B): $= \oint_{x^2+y^2=a^2,\ 0<a<1} M\,dx + N\,dy = 2\pi$

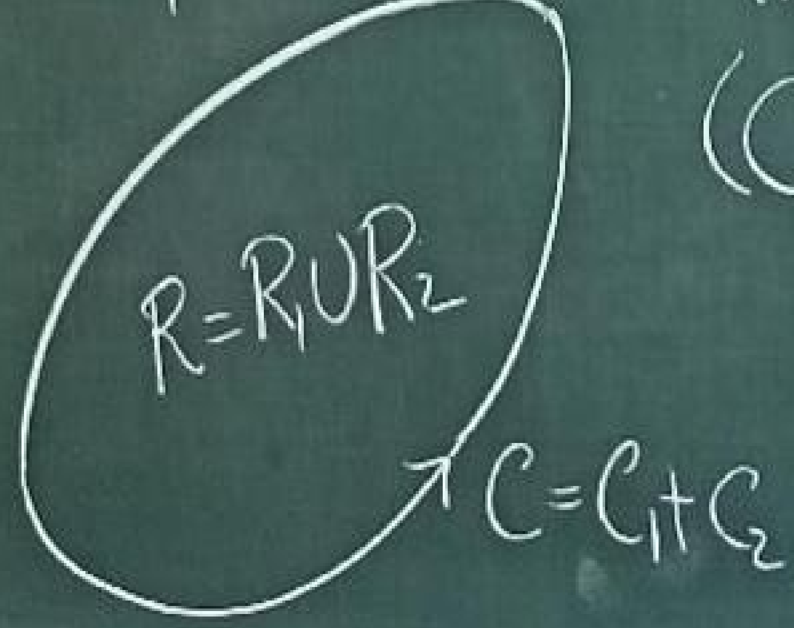
Pf of Green's Theorem (tangential form)

$$(i)\ \oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

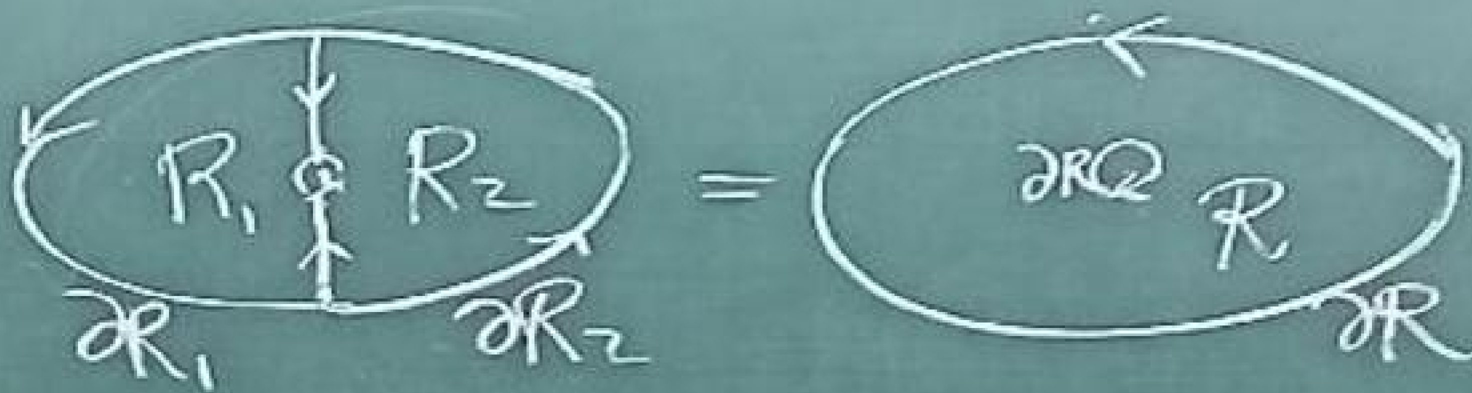
(pf for normal form is similar)



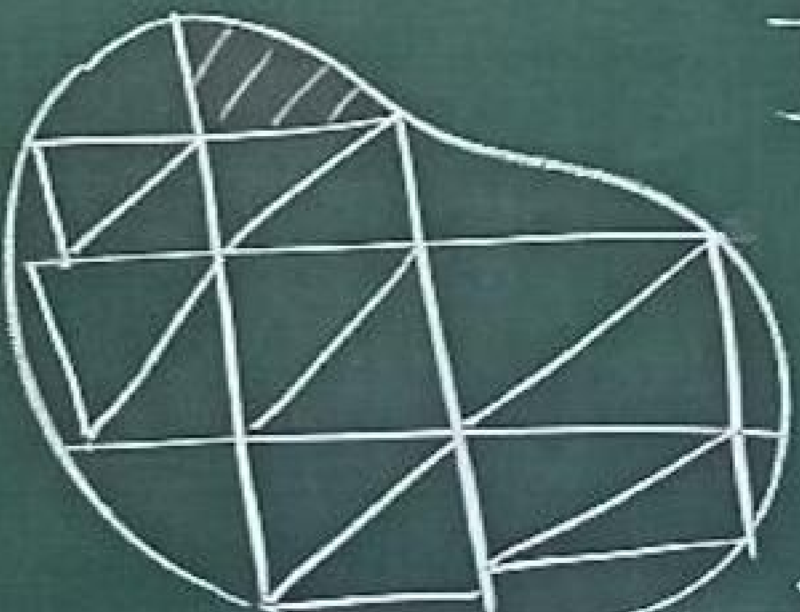
If (i) is true for (C_1, R_1), (C_2, R_2) then it is true for (C, R) where
$C = C_1 + C_2$
$R = R_1 \cup R_2$



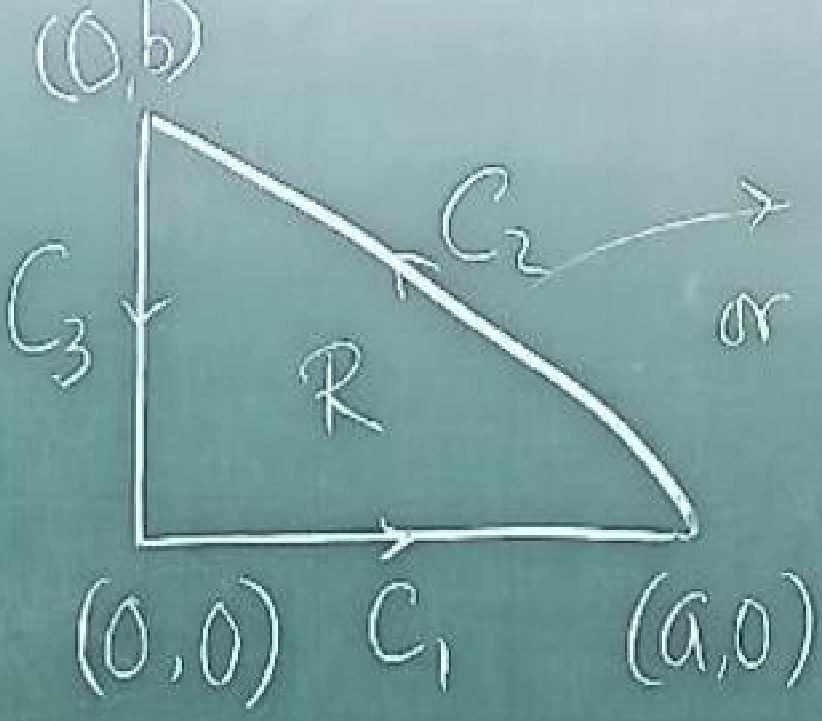
Rm in previous example



Similarly



In general, it suffice to verify R of the form 



$y = f(x) \quad 0 \le x \le a$

or $x = g(y) \quad 0 \le y \le b$

$$C_2 = \{(x, f(x)),\ 0 \le x \le a\} = \{(g(y), y),\ 0 \le y \le b\}$$

Check tangential form, line integrals:

$$\int_{C_1} \underset{\vec{T}_1=(1,0)}{\overset{t=x}{=}} \int_{t=0}^{a} \vec{F}\cdot(1,0)\,dt = \int_0^a M(t,0)\,dt = \int_0^a M(x,0)\,dx \quad (\mathrm{I})$$

$$\int_{C_3} \underset{\vec{T}=(0,-1)}{\overset{t=b-y}{=}} \int_{t=0}^{b} -N(0,b-t)\,dt = -\int_{y=b}^{0} N(0,y)(-dy)$$

$$\|$$

$$\text{or } \int_{C_3\downarrow} = -\int_{-C_3\uparrow} \underset{\vec{T}=(0,1)}{\overset{s=y}{=}} -\int_{s=0}^{b} N(0,s)\,ds = -\int_{y=0}^{b} N(0,y)\,dy \quad (\mathrm{III})$$

$$\int_{C_2} \underset{t=y}{=} \int_{t=0}^{b} \left(M\frac{dx}{dt} + N\frac{dy}{dt}\right)dt = \int_0^b \Big(\underbrace{M(g(t),t)\,g'(t)}_{(\mathrm{II}_1)} + \underbrace{N(g(t),t)}_{(\mathrm{II}_2)}\Big)dt$$

$C_2 = \{y=t,\ x=g(t),\ 0\le t\le b\}$ $\quad \dot{x} = g'(t)$, $\dot{y}=1$

Double integrals $\iint_R (N_x - M_y)\,dA$

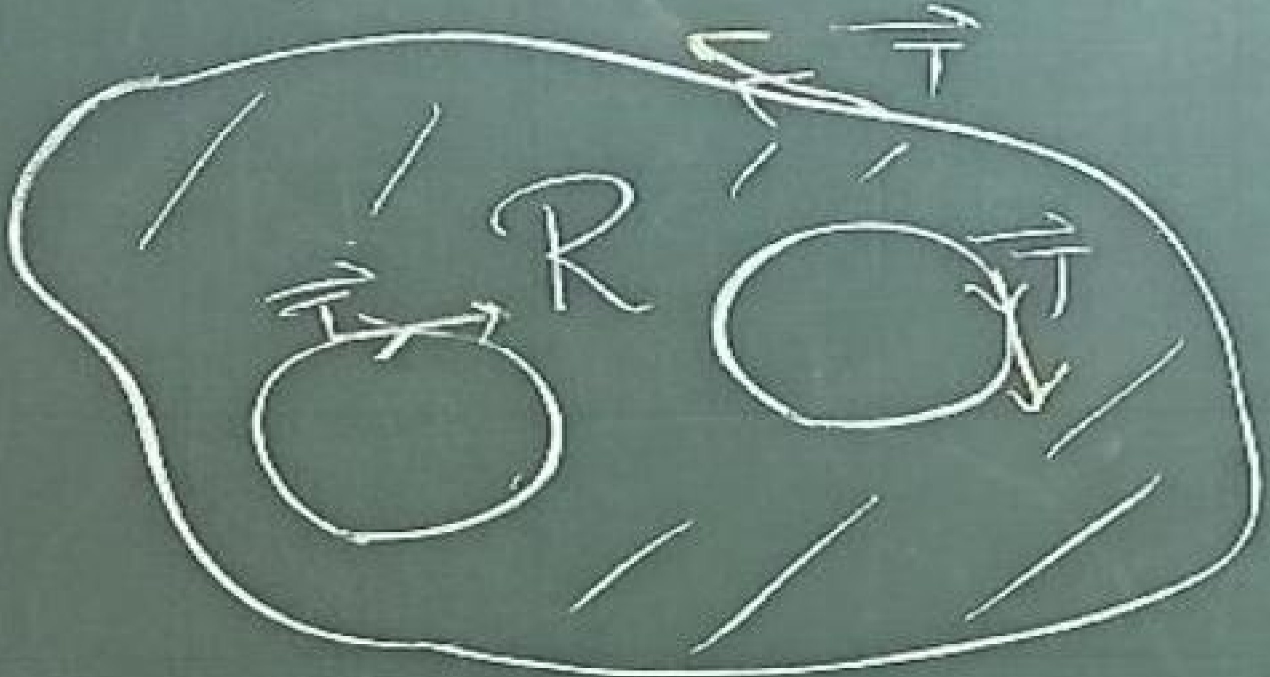
$$\iint_R N_x\,dA = \int_0^b\int_0^{g(y)} N_x\,dx\,dy = \int_0^b \Big(\underbrace{N(g(y),y)}_{(\mathrm{II}_2)} - \underbrace{N(0,y)}_{(\mathrm{III})}\Big)dy$$

$$-\iint_R M_y\,dA = -\int_0^a \int_{y=0}^{f(x)} M_y\,dy\,dx = -\int_0^a \Big(\underbrace{M(x,f(x))}_{(\mathrm{IV})} - \underbrace{M(x,0)}_{(\mathrm{I})}\Big)dx$$

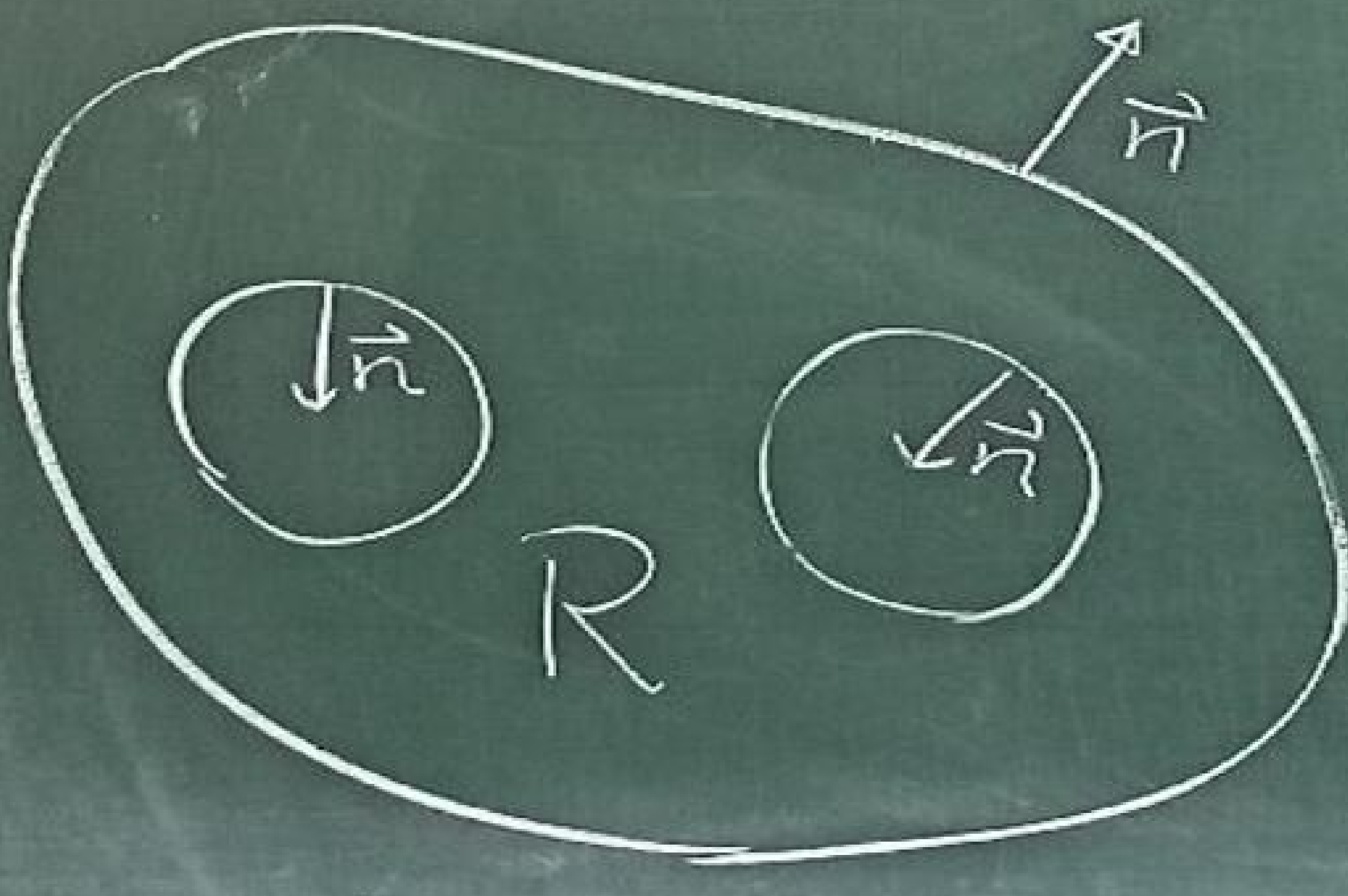
$(\mathrm{IV}) = -\int_0^a M(x,f(x))\,dx$ $\quad y=f(x)$, $x=f^{-1}(y)\equiv g(y)$

$= -\underbrace{\int_{y=b}^{0} M(g(y),y)\,g'(y)\,dy}_{=(\mathrm{II}_1)}$ $\quad dx = g'(y)\,dy$ $\quad \therefore$ (i) verified

Tangential form



Normal form



$\underline{\underline{\text{Rm}}}$ For $\vec{F} = \dfrac{(-y, x)}{x^2+y^2}$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \begin{cases} 2\pi, & (0,0) \in R \\ 0, & (0,0) \notin R \end{cases}$$

$R =$ inside of C

$\left[\underline{\underline{\text{Rm}}}\ \dfrac{(-y, x)}{x^2+y^2} = \nabla \tan^{-1}\left(\dfrac{y}{x}\right) \right.$

But $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ is NOT

$\left. \text{everywhere defined on } x^2+y^2 \neq 0 \right]$