Summary of Section 16.3

$$\begin{array}{lcl}
\mathbf{F} = \nabla f \text{ on } D & & \\
\Big\Updownarrow \text{ Theorem 2} & & \\
\mathbf{F} \text{ conservative on } D & \overset{\nabla\times\nabla f = \mathbf{0}}{\Longrightarrow} & \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D \\
\left(\displaystyle\int_A^B \mathbf{F}\cdot d\mathbf{r} \text{ is path independent in } D \right) & & \text{(i.e. } \mathbf{F} \text{ satisfies the component test)} \\
\Big\Updownarrow \text{ Theorem 3} & & (\Longleftarrow \text{ only if } D \text{ is simply connected}) \\
\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0 & & \\
\text{over any closed path in } D & &
\end{array}$$

Note:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\mathbf{k}$$

$\mathbf{F} = \nabla f \Longrightarrow M = f_x,\ N = f_y,\ P = f_z$ satisfies $P_y = N_z,\ M_z = P_x,\ N_x = M_y$

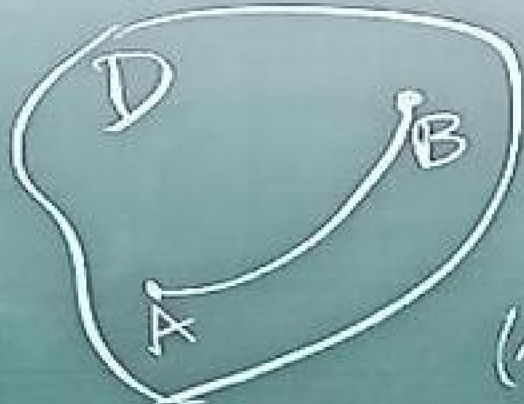
Thm 2: Let D be an open connected domain, $\vec{F}: D \to \mathbb{R}^3$ cont.

Then "$\vec{F}$ is conservative in D"

$\Leftrightarrow$ "$\vec{F} = \nabla f$ for some diff. function f in D".

Pf: "$\Leftarrow$": Theorem 1.

"$\Rightarrow$": We construct this f explicitly as follows.



Take any point $A \in D$

(i) Define $f(A) = 0$

(ii) For any $B \in D$

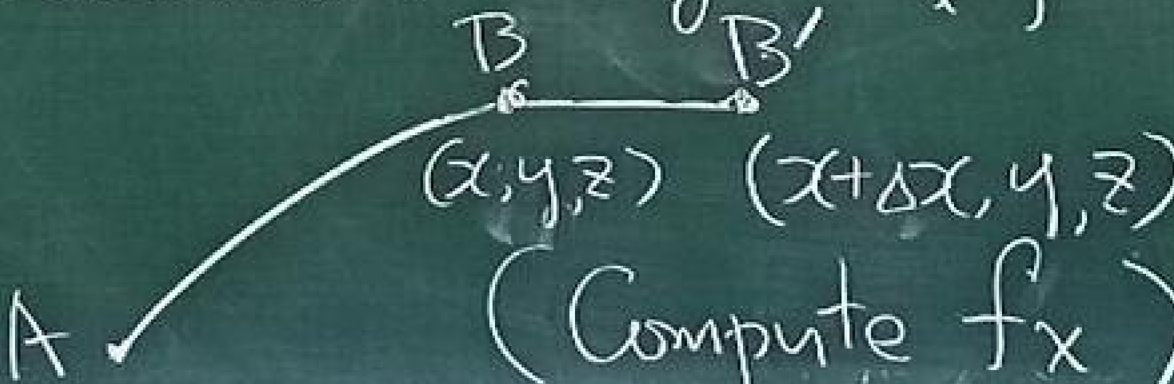
define $f(B) = \int_C \vec{F} \cdot \vec{T}\, ds$

where C is <u>any</u> curve from A to B

$\Rightarrow$ f is defined everywhere in D.

(Note $f(B)$ is independent of C)

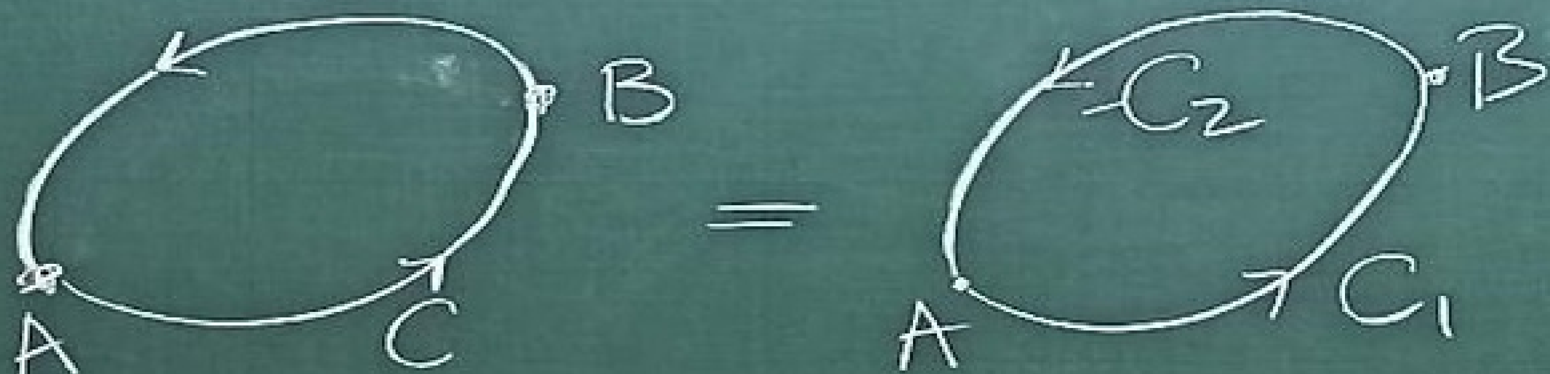
Question: Why $\nabla f = \vec{F}$?



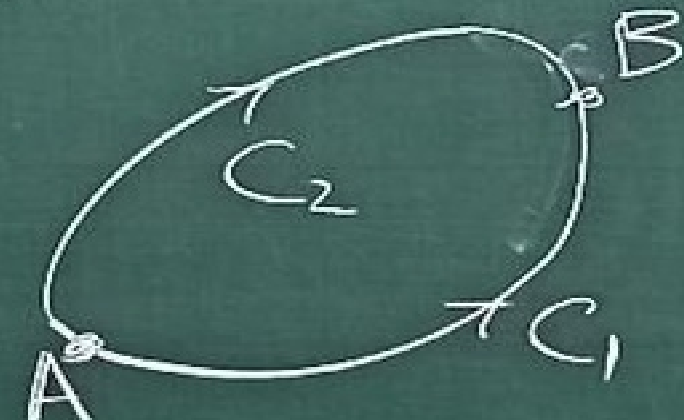
(Compute f_x)

Thm 3: $\vec{F}$ is conservative in D

$\Leftrightarrow \int_C \vec{F}\cdot\vec{T}\,ds = 0$ for any closed curve $C \subseteq D$



$\int_C = \int_{C_1} + \int_{-C_2}$



$= \int_{C_1} - \int_{C_2}$

$\therefore \int_C = 0 \Leftrightarrow \int_{C_1} = \int_{C_2}$

$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{A\to B\to B'} \vec{F}\cdot\vec{T}\,ds - \int_{A\to B} \vec{F}\cdot\vec{T}\,ds}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_{B\to B'} \vec{F}\cdot\vec{T}\,ds}{\Delta x}$$

$$\left(B \to B' : \vec{T} = (1,0,0),\ \vec{F}\cdot\vec{T} = F_1 \right)$$

$$= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} F_1(t,y,z)\,dt}{\Delta x} = F_1(x,y,z)$$

i.e. $\overline{BB'} = \{(t,y,z).\ x \leq t \leq x+\Delta x\}$

$\therefore f_x(x,y,z) = F_1(x,y,z)$ Similarly

$f_y = F_2,\ f_z = F_3 \ \therefore \nabla f = \vec{F}$

Given a vector field $\vec{F}$, how to determine if $\vec{F}$ is conservative?

Step 1: Apply the Component Test

For 2D Vector Fields $\vec{F} = \langle M, N \rangle$:

1. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\vec{F}$ passes the 2D component test.
2. **If** $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$: $\vec{F}$ is not conservative.

For 3D Vector Fields $\vec{F} = \langle M, N, P \rangle$:

1. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$, $\vec{F}$ passes the 3D component test.
2. **If** $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ **or** $\frac{\partial M}{\partial z} \neq \frac{\partial P}{\partial x}$ **or** $\frac{\partial N}{\partial z} \neq \frac{\partial P}{\partial y}$: $\vec{F}$ is not conservative.

Step 2: If $\vec{F}$ passes the component test, check the domain

1. **If the domain is open and simply connected:** $\vec{F}$ is guaranteed to be conservative (Green's Theorem). Proceed to Step 3.
2. **If the domain is not simply connected:** The component test is not enough. $\vec{F}$ may or may not be conservative.
   1. To prove it **is conservative**: You must directly find a global potential function f.
   2. To prove it **is not conservative**: Show that a line integral around a closed loop enclosing a hole does not equal zero.

Step 3: Try to find the potential function

Question:

When is $\vec{F}$ conservative?
i.e. Can we find the potential f?

Eg 1. $\vec{F} = (e^x \cos y + yz, xz - e^x \sin y, xy + z)$

Is $\vec{F}$ conservative?

Sol: Check component test first

Is $(*)$ true? No $\Rightarrow$ Not conservative
(component test) Yes $\Rightarrow$ try to find f.

Check. $M_y = -e^x \sin y + z = N_x$

$N_z = x = P_y$

$P_x = y = M_z$

(component test)

(*) holds $\Rightarrow$ f may probably exist!

How to find f ?

$$f_x = e^x \cos y + yz$$

$$\Rightarrow f = \int (e^x \cos y + yz)\,dx$$

$$= e^x \cos y + xyz + C_1(y,z)$$

Similarly

$$f = xyz + e^x \cos y + C_2(x,z)$$

$$f = xyz + \frac{z^2}{2} + C_3(x,y)$$

$$\Rightarrow f(x,y,z)$$

$$= xyz + e^x \cos y + \frac{z^2}{2}$$ will do!

Eg2. $\vec{F}=\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

Is $\vec{F}$ conservative?

Sol: $M_y = N_x = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore$ $(*)$ holds!

(component test)

$\vec{F}$ is defined on $D=\mathbb{R}^3 \setminus z$ axis

D is not simply connected.

Try to find $f \Longrightarrow$ does not work

Try to prove $\vec{F}$ is not conservative.

$\Longrightarrow$ Use Thm 2

Take C: $\vec{r}(t)=(a\cos t, a\sin t, 0)$

with $a>0$ fixed, $0\le t\le 2\pi$

$\dot{\vec{r}}(t)=(-a\sin t, a\cos t, 0)$

$$\oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{a}, \frac{\cos t}{a}, 0\right)\cdot(-a\sin t, a\cos t, 0)\,dt$$

$$= 2\pi$$

From Thm 2 $\Rightarrow$ $\vec{F}$ is not conservative.

Eg 3: $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$ conservative?

Sol: $N_y = M_x = \frac{-1}{2}\left(xy(x^2+y^2)^{-\frac{3}{2}}\right)$

$\therefore$ f may exist. Take same C as in Eg 2.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (c, s, 0)\cdot(-as, ac, 0)\,dt = 0$$

Try to find f $\Rightarrow$ $f(x,y) = \sqrt{x^2+y^2} + C$

$\therefore$ $\vec{F}$ is conservative!