

Summary of Section 16.3

$$\mathbf{F} = \nabla f \text{ on } D$$

\Updownarrow Theorem 2

\mathbf{F} conservative on D

$$\left(\int_A^B \mathbf{F} \cdot d\mathbf{r} \text{ is path independent in } D \right)$$

\Updownarrow Theorem 3

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

over any closed path in D

$$\nabla \times \nabla f = \mathbf{0} \implies \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D$$

(i.e. \mathbf{F} satisfies the component test)

(\Leftarrow only if D is simply connected)

Note:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

$$\mathbf{F} = \nabla f \implies M = f_x, N = f_y, P = f_z \text{ satisfies } P_y = N_z, M_z = P_x, N_x = M_y$$

Thm 2: Let D be an open
connected domain, $\vec{F}: D \rightarrow \mathbb{R}^3$ cont.

Then \vec{F} is conservative in D

$\Leftrightarrow \vec{F} = \nabla f$ for some diff.
function f in D .

pf: " \Leftarrow ". Theorem **1**.

" \Rightarrow ": We construct this f
explicitly as follows.



Take any point $A \in D$

(i) Define $f(A) = 0$

(ii) For any $B \in D$

define $f(B) = \int_C \vec{F} \cdot \vec{T} ds$

where C is any curve from A to B

$\Rightarrow f$ is defined everywhere in D .

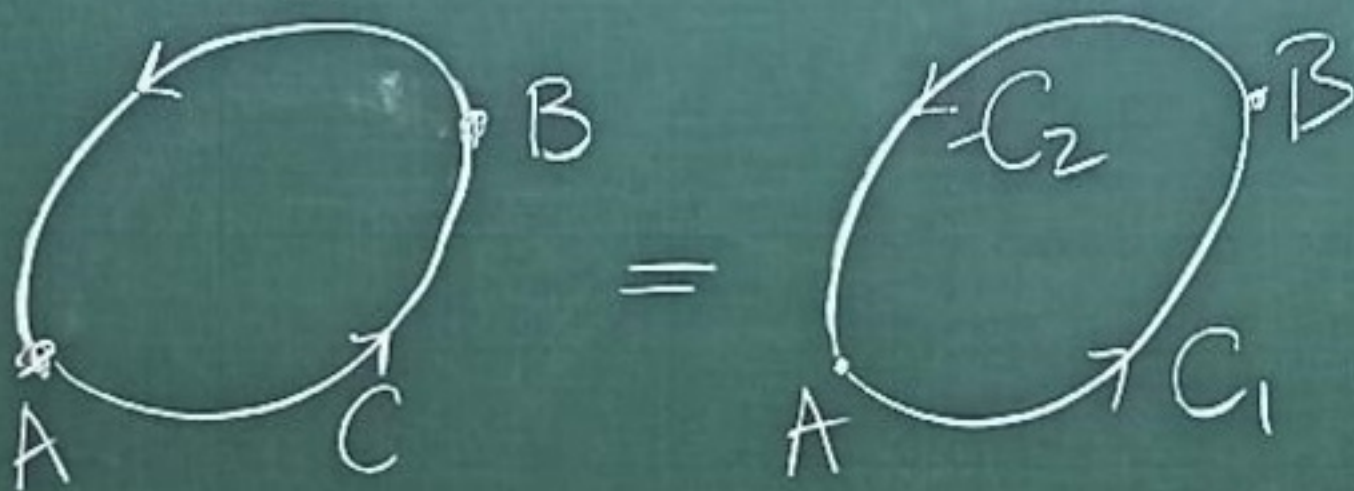
(Note $f(B)$ is independent of C)

Question: Why $\nabla f = \vec{F}$?



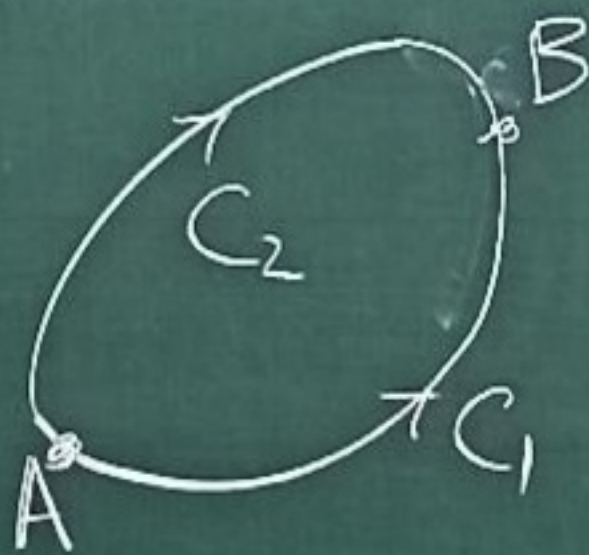
Thm 3: \vec{F} is conservative in D

$\Leftrightarrow \int_C \vec{F} \cdot \vec{T} ds = 0$ for any closed curve $C \subseteq D$



$$\int_C = \int_{C_1} + \int_{-C_2}$$

$$= \int_{C_1} - \int_{C_2}$$



$$\therefore \int_C = 0 \Leftrightarrow \int_{C_1} = \int_{C_2}$$

$$f_x(x, y, z) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\int_{A \rightarrow B \rightarrow B'} \vec{F} \cdot \vec{T} ds - \int_{A \rightarrow B} \vec{F} \cdot \vec{T} ds}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\int_{B \rightarrow B'} \vec{F} \cdot \vec{T} ds}{\Delta x}$$

$$(B \rightarrow B': \vec{T} = (1, 0, 0), \vec{F} \cdot \vec{T} = F_1)$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\int_x^{x+\Delta x} F_1(t, y, z) dt}{\Delta x} = F_1(x, y, z)$$

$$\text{i.e. } \overline{B \rightarrow B'} = \{(t, y, z), x \leq t \leq x + \Delta x\}$$

$$\therefore f_x(x, y, z) = F_1(x, y, z) \text{ Similarly}$$

$$f_y = F_2, f_z = F_3 \therefore \nabla f = \vec{F}$$

Given a vector field \vec{F} , how to determine if \vec{F} is conservative?

Step 1: Apply the Component Test

For 2D Vector Fields $\vec{F} = \langle M, N \rangle$:

1. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, \vec{F} passes the 2D component test.
2. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$: \vec{F} is not conservative.

For 3D Vector Fields $\vec{F} = \langle M, N, P \rangle$:

1. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$, \vec{F} passes the 3D component test.
2. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ or $\frac{\partial M}{\partial z} \neq \frac{\partial P}{\partial x}$ or $\frac{\partial N}{\partial z} \neq \frac{\partial P}{\partial y}$: \vec{F} is not conservative.

Step 2: If \vec{F} passes the component test, check the domain

1. **If the domain is open and simply connected:** \vec{F} is guaranteed to be conservative (Green's Theorem). Proceed to Step 3.
2. **If the domain is not simply connected:** The component test is not enough. \vec{F} may or may not be conservative.
 1. To prove it **is conservative**: You must directly find a global potential function f .
 2. To prove it **is not conservative**: Show that a line integral around a closed loop enclosing a hole does not equal zero.

Step 3: Try to find the potential function

Question:

When is \vec{F} conservative?

i.e. Can we find the potential f ?

Eg 1. $\vec{F} = (e^x \cos y + yz, xz - e^x \sin y, xy + z)$

Is \vec{F} conservative?

Sol: Check component test first

Is (*) true? No \Rightarrow Not conservative
(component test) Yes \Rightarrow try to find f .

check:

$$\begin{aligned} M_y &= -e^x \sin y + z = N_x \\ N_z &= x = P_y \\ P_x &= y = M_z \end{aligned}$$

(component test)

(*) holds $\Rightarrow f$ may probably exist!

How to find f ?

$$f_x = e^x \cos y + yz$$

$$\Rightarrow f = \int (e^x \cos y + yz) dx$$

$$= e^x \cos y + xyz + C_1(y, z)$$

Similarly

$$f = xyz + e^x \cos y + C_2(x, z)$$

$$f = xyz + \frac{z^2}{2} + C_3(x, y)$$

$$\Rightarrow f(x, y, z)$$

$$= xyz + e^x \cos y + \frac{z^2}{2} \text{ will do!}$$

$$\text{Ex 2. } \vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

Is \vec{F} conservative?

$$\text{Sol: } M_y = N_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

(*) holds!
(component test)

\vec{F} is defined on $D = \mathbb{R}^3 \setminus z \text{ axis}$

D is not simply connected.

Try to find $f \Rightarrow$ does not work

Try to prove \vec{F} is not conservative.

\Rightarrow Use Thm 2

Take $C: \vec{r}(t) = (a \cos t, a \sin t, 0)$

with $a > 0$ fixed, $0 \leq t \leq 2\pi$

$$\dot{\vec{r}}(t) = (-a \sin t, a \cos t, 0)$$

$$\oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{a}, \frac{\cos t}{a}, 0 \right) \cdot (-a \sin t, a \cos t, 0) dt$$

$$= 2\pi$$

From Thm 2 $\Rightarrow \vec{F}$ is not conservative.

Ex 3: $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right)$ conservative?

Sol. $N_y = M_x = \frac{1}{2} (xy (x^2+y^2)^{-\frac{3}{2}})$

$\therefore f$ may exist. Take same C as in Ex 2.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (c, s, 0) \cdot (-as, ac, 0) dt = 0$$

Try to find $f \Rightarrow f(x, y) = \sqrt{x^2+y^2} + C$

$\therefore \vec{F}$ is conservative!