

Fundamental Theorem of line integral

Thm 1 If f is differentiable in D

, ∇f is continuous in D and

C is any curve in D

from point A to point B , then

$$\int_C \nabla f \cdot \vec{T} ds = f(B) - f(A)$$

If Let $C = \{ \vec{r}(t), 0 \leq t \leq 1 \}$

with $A = \vec{r}(0)$, $B = \vec{r}(1)$

$$\begin{aligned} \Rightarrow \int_C \nabla f \cdot \vec{T} ds &= \int_0^1 \left(f_x(x(t), y(t), z(t)) \dot{x}(t) \right. \\ &\quad \left. + f_y(x(t), y(t), z(t)) \dot{y}(t) \right. \\ &\quad \left. + f_z(x(t), y(t), z(t)) \dot{z}(t) \right) dt \\ &= \int_0^1 \frac{d}{dt} f(x(t), y(t), z(t)) dt \\ &= f(x(t), y(t), z(t)) \Big|_{t=0}^1 = f(B) - f(A) \end{aligned}$$

Ex 5: Evaluate $\int_A^B \vec{F} \cdot \vec{T} ds$
along C , where $A = (2, 0)$

$$B = (-2, 0) \quad C = \left\{ \frac{x^2}{4} + y^2 = 1, y \geq 0 \right\}$$

$$\vec{F} = \frac{(x, y)}{\sqrt{x^2 + y^2}}$$

Sol. $\vec{F} = \nabla(\sqrt{x^2 + y^2})$

$$\Rightarrow \int_A^B \vec{F} \cdot \vec{T} ds$$

$$= \int_A^B \nabla(\sqrt{x^2 + y^2}) \cdot \vec{T} ds$$

$$= \sqrt{x^2 + y^2} \Big|_{\substack{(-2, 0) = B \\ (2, 0) = A}} = 0$$

Remark: If $\vec{F} = \nabla f$

Then $\int_C \vec{F} \cdot \vec{T} ds = 0$

for any closed curve C

Def. D is "connected"

If any two point in D
can be connected by a
path (curve) in D



connected

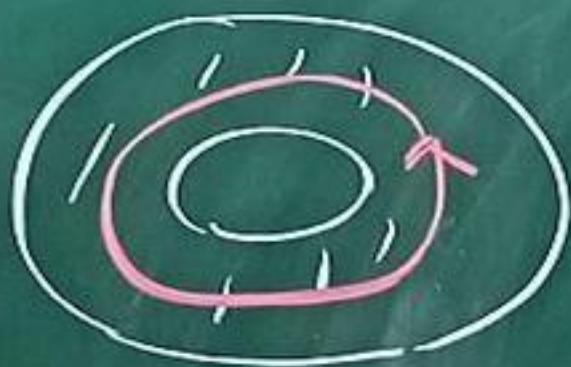


Not connected

Def D is Simply connected

If any closed curve in D
can be continuously shrunk
to a point in D without leaving D

Eg



$$D = \{(x, y), 1 \leq x^2 + y^2 \leq 4\}$$

Not Simply connected



: No

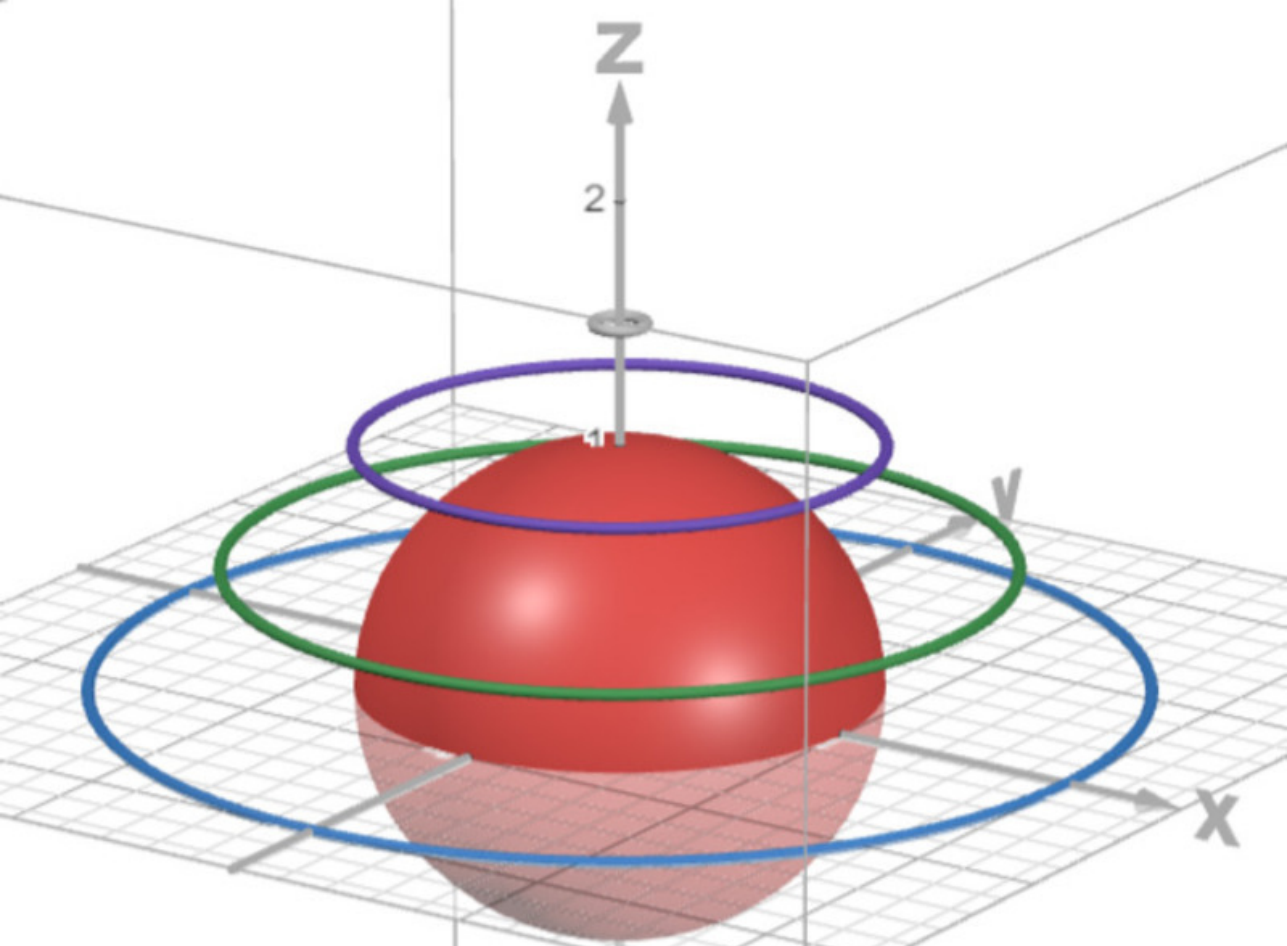
$$D = \mathbb{R}^2 - \{(0,0)\}$$

No

$$D = \{(x, y, z), 1 \leq x^2 + y^2 \leq 4\} : \text{No}$$

$$D = \{(x, y, z), 1 \leq x^2 + y^2 + z^2 \leq 4\}, \text{Yes}$$

$$D = \mathbb{R}^3 - \{(0,0,0)\} : \text{Yes}$$



$\{p > 1\}$ is simply connected

單連通 (Simply Connected) 區域範例

區域代數表達式 (D)	幾何圖形描述	是否為單連通?
$x^2 + y^2 + z^2 < 4$	開放實心球體	是 (Yes)
$1 < x^2 + y^2 + z^2 < 4$	開放空心球殼	是 (Yes)
$1 < x^2 + y^2 + z^2$	開放實心球體外部空間	是 (Yes)
$0 < x^2 + y^2 + z^2 < 4$	開放實心球體挖去原點	是 (Yes)
$0 < x^2 + y^2 + z^2$	三維空間挖去原點	是 (Yes)
$1 < x^2 + y^2 < 4$ in \mathbb{R}^3	中空無限長圓柱體	否 (No)
$1 < x^2 + y^2$ in \mathbb{R}^3	開放圓柱體外部空間	否 (No)
$0 < x^2 + y^2 < 4$ in \mathbb{R}^3	開放圓柱體挖去中心軸	否 (No)
$0 < x^2 + y^2$ in \mathbb{R}^3	三維空間挖去整個 z 軸	否 (No)
$(\sqrt{x^2 + y^2} - 2)^2 + z^2 < 1$	實心回環 (甜甜圈形狀)	否 (No)