Fundamental Theorem of line integral

Thm 1 If f is differentiable in D, ∇f is continuous in D and C is <u>any</u> curve in D <u>from</u> point A <u>to</u> point B, then

$$\int_C \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$$

Pf Let $C = \{\vec{r}(t),\ 0 \le t \le 1\}$ with $A = \vec{r}(0)$, $B = \vec{r}(1)$

$$\Rightarrow \int_C \nabla f \cdot \vec{T}\, ds = \int_0^1 \begin{pmatrix} f_x(x(t), y(t), z(t))\, \dot{x}(t) \\ + f_y(x(t), y(t), z(t))\, \dot{y}(t) \\ + f_z(x(t), y(t), z(t))\, \dot{z}(t) \end{pmatrix} dt$$

$$= \int_0^1 \frac{d}{dt} f(x(t), y(t), z(t))\, dt$$

$$= f(x(t), y(t), z(t)) \Big|_{t=0}^{1} = f(B) - f(A)$$

Eg 5: Evaluate $\int_A^B \vec{F} \cdot \vec{T} \, ds$

along C, where $A = (2,0)$

$B = (-2,0)$ $C = \{ \frac{x^2}{4} + y^2 = 1, \ y \geq 0 \}$

$$\vec{F} = \frac{(x, y)}{\sqrt{x^2+y^2}}$$

Sol: $\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$

$$\Rightarrow \int_A^B \vec{F} \cdot \vec{T} \, ds$$

$$= \int_A^B \nabla\left(\sqrt{x^2+y^2}\right) \cdot \vec{T} \, ds$$

$$= \sqrt{x^2+y^2} \Big|_{\underset{A}{(2,0)}}^{(-2,0)=B} = 0$$

Remark: If $\vec{F} = \nabla f$

Then $\int_C \vec{F} \cdot \vec{T} \, ds = 0$

for any closed curve C

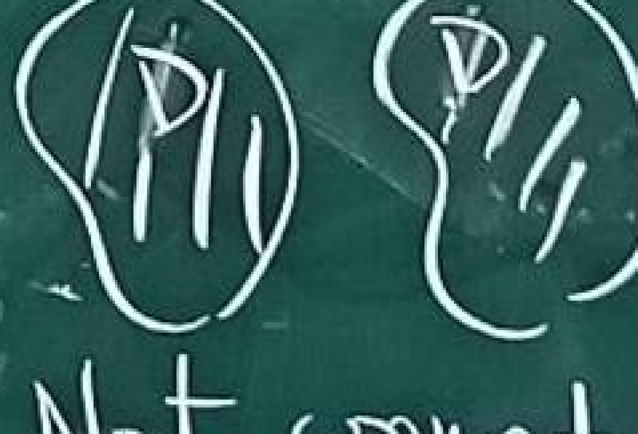
Def. D is "connected"

If any two point in D

can be connected by a

path (curve) in D

Connected

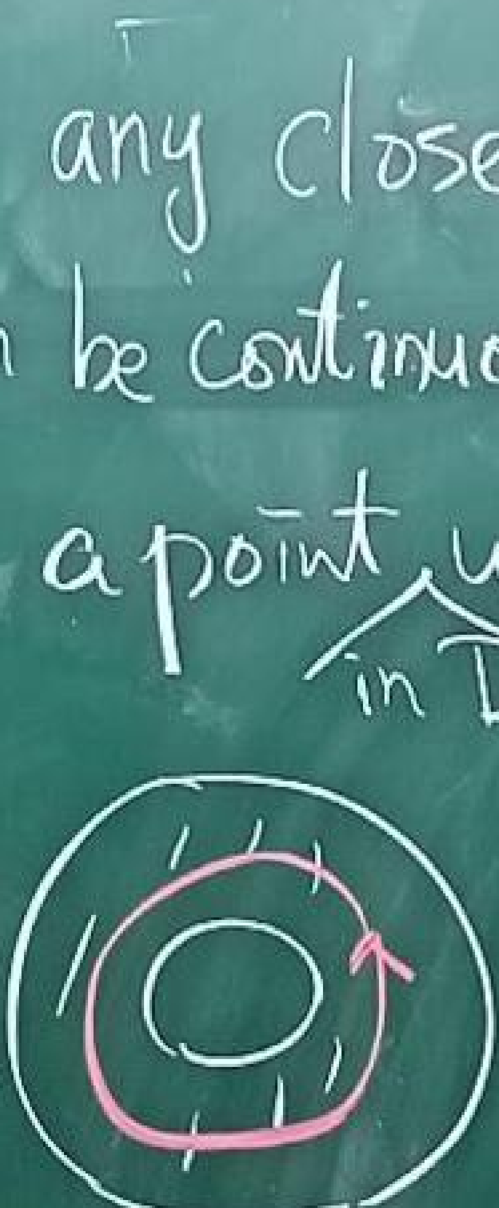


Not connected

Def. D is simply connected

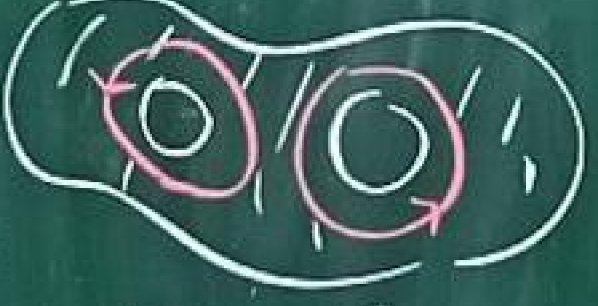
If any closed curve in D can be continuously shrinked to a point in D without leaving D

Eg



$D=\{(x,y),\ 1\le x^2+y^2\le 4\}$

Not simply connected



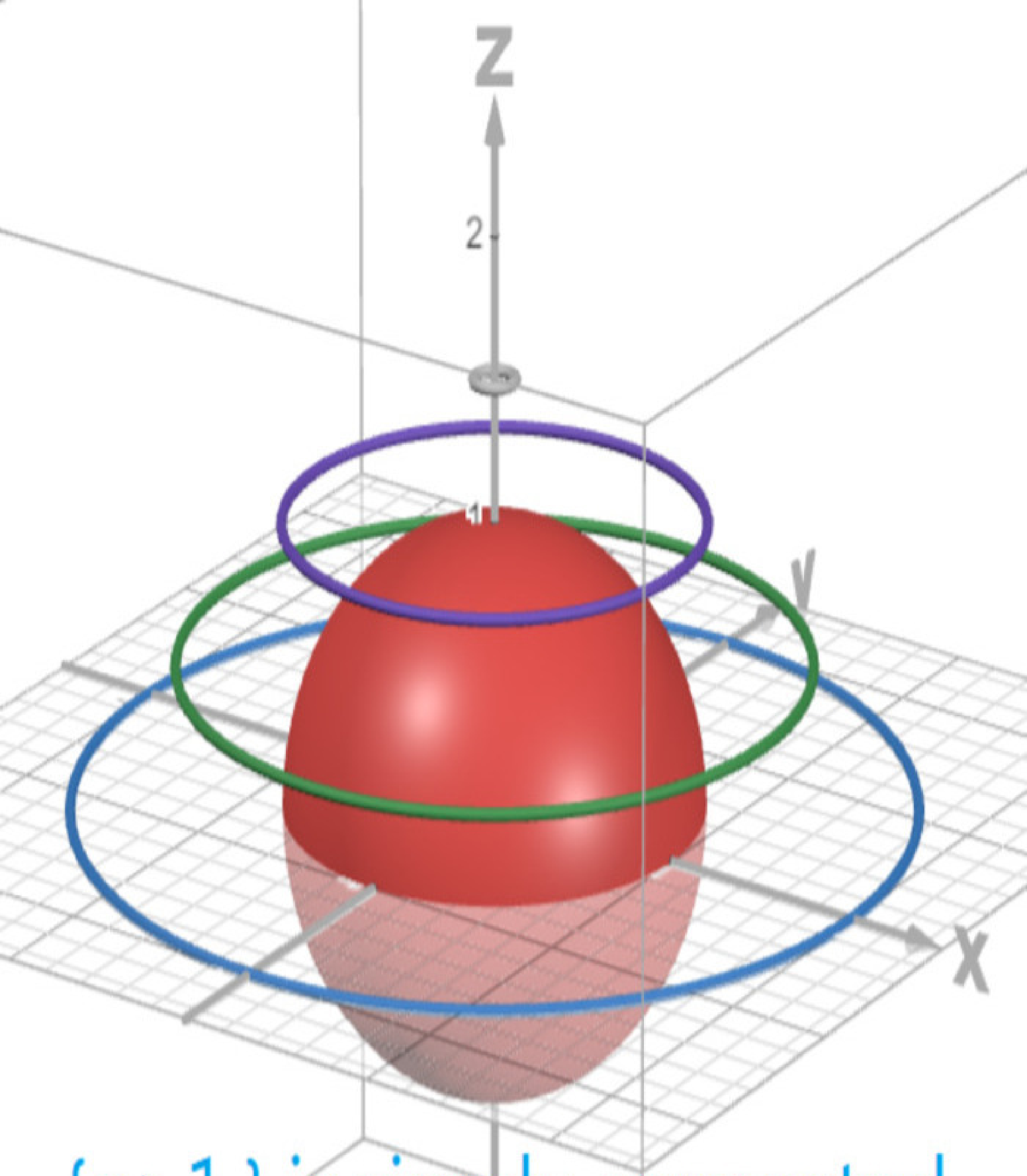
: No

$\boxed{\begin{array}{c} D=\mathbb{R}^2\setminus\{(0,0)\} \\ \text{No} \end{array}}$

$D=\{(x,y,z)\ 1\le x^2+y^2\le 4\}$: No

$D=\{(x,y,z),\ 1\le x^2+y^2+z^2\le 4\}$, Yes

$D=\mathbb{R}^3\setminus\{(0,0,0)\}$: Yes



$\{\rho > 1\}$ is simply connected

單連通 (Simply Connected) 區域範例

區域代數表達式 (D)	幾何圖形描述	是否為單連通？
$x^2 + y^2 + z^2 < 4$	開放實心球體	是 (Yes)
$1 < x^2 + y^2 + z^2 < 4$	開放空心球殼	是 (Yes)
$1 < x^2 + y^2 + z^2$	開放實心球體外部空間	是 (Yes)
$0 < x^2 + y^2 + z^2 < 4$	開放實心球體挖去原點	是 (Yes)
$0 < x^2 + y^2 + z^2$	三維空間挖去原點	是 (Yes)
$1 < x^2 + y^2 < 4$ in $\mathbb{R}^3$	中空無限長圓柱體	否 (No)
$1 < x^2 + y^2$ in $\mathbb{R}^3$	開放圓柱體外部空間	否 (No)
$0 < x^2 + y^2 < 4$ in $\mathbb{R}^3$	開放圓柱體挖去中心軸	否 (No)
$0 < x^2 + y^2$ in $\mathbb{R}^3$	三維空間挖去整個 z 軸	否 (No)
$(\sqrt{x^2 + y^2} - 2)^2 + z^2 < 1$	實心圓環（甜甜圈形狀）	否 (No)