

Line integrals

Goal $\int_{\vec{P}}^{\vec{Q}} \nabla f(\vec{r}) \cdot \underbrace{\vec{t}}_{d\vec{r}} ds = f(\vec{Q}) - f(\vec{P})$

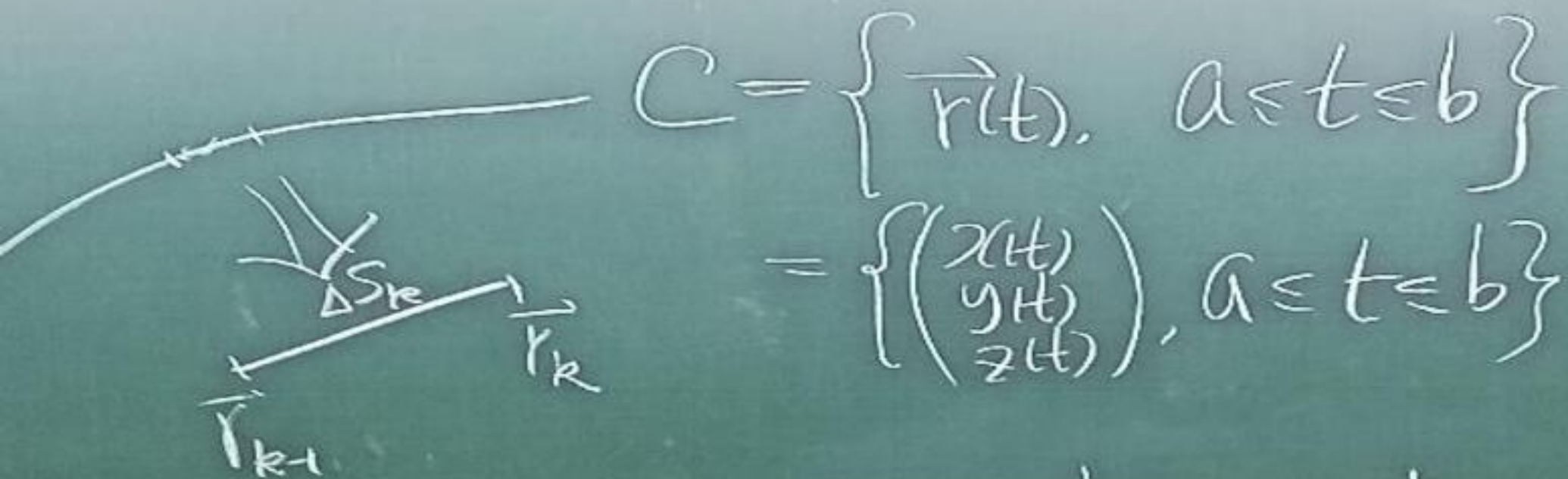
$(\vec{P}, \vec{Q} \in \mathbb{R}^3 \quad f: \mathbb{R}^3 \rightarrow \mathbb{R})$

We start with a related integral

$$\int_C f(x, y, z) ds = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta S_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.



$$\vec{r}_k = \vec{r}(t_k)$$

$$\Delta t_k = t_k - t_{k-1}$$

$$\Delta S_k = |\vec{r}_k - \vec{r}_{k-1}|$$

$$= \frac{|\vec{r}_k - \vec{r}_{k-1}|}{\Delta t_k} \Delta t_k$$

$$\therefore dS = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$

Line integrals

Eg 1. Evaluate $\int_C f(x, y, z) ds$

where $f = x - 3y^2 + z$

C = line segment between
 $(0, 0, 0)$ and $(1, 1, 1)$.

Sol: Step 1: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t, t, t)$, $0 \leq t \leq 1$

Step 2: $ds = |\vec{r}'(t)| dt = |(1, 1, 1)| dt$

Step 3: $\text{Ans} = \int_{t=0}^1 (t - 3t^2 + t) \sqrt{3} dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

Rm: the value of $I = \int_C f(x, y, z) ds$,
is independent of the parameter
 $\vec{r}(t)$. For example

$$\vec{r}(t) = (t^2, t^2, t^2), \quad 0 \leq t \leq 1$$

$$\text{or } \vec{r}(t) = (1-t, 1-t, 1-t), \quad 0 \leq t \leq 1$$

all give the same value of

I as long as $\vec{r}(t)$ is
a correct parametrization
for C .

$$\text{Eq 2: } f(x, y, z) = x - 3y^2 + z$$

$$C = C_1 \cup C_2 \quad (\text{two line segments})$$

$$C_1 = \overline{(0, 0, 0) \quad (1, 1, 0)}$$

$$C_2 = \overline{(1, 1, 0) \quad (1, 1, 1)}$$

Sol: $C_1: \vec{r}_1(t) = (t, t, 0), 0 \leq t \leq 1$

$$C_2: \vec{r}_2(t) = (1, 1, t), 0 \leq t \leq 1$$

$$S_C = S_{C_1} + S_{C_2} \quad \left| \frac{d\vec{r}_1}{dt} \right| \quad \left| \frac{d\vec{r}_2}{dt} \right|$$

$$= \int_0^1 (t - 3t^2) \sqrt{2} dt + \int_0^1 (-2t) \cdot 1 \cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2} \quad \left(\begin{array}{l} \text{Compare with Eq 1, same } f \\ \text{different } C, \text{ (same end points)} \\ \rightarrow \text{different answers} \end{array} \right)$$

Related integral

$$(2) \int_C \vec{F} \cdot \vec{T} \, ds$$

$$\vec{F}: D \longrightarrow \mathbb{R}^3$$

$(x, y, z) \quad (F_1(x, y, z), F_2(\cdot), F_3(\cdot))$

C : a smooth curve
with prescribed orientation (指向)

$\vec{r}(t)$: parametrization of C with

"direction of increasing t "

= "orientation of C "

$$\vec{T} = \frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|}$$

Eg 3 Evaluate $\int_C \vec{F} \cdot \vec{T} ds$

where $\vec{F} = (z, xy, -y^2)$

$$= \int_C \vec{F} \cdot d\vec{F} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$C: \vec{r}(t) = (t^2, t, \sqrt{t}), 0 \leq t \leq 1$

Sol. $\vec{r}(t) = (2t, 1, \frac{1}{2\sqrt{t}})$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_{t=0}^1 \vec{F} \cdot \frac{\vec{r}(t)}{|\vec{r}(t)|} |\dot{\vec{r}}(t)| dt$$

$$= \int_0^1 (\sqrt{t}, t^3, -t^2) \cdot (2t, 1, \frac{1}{2\sqrt{t}}) dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2} \right) dt$$

$$= \frac{3}{2} \cdot \frac{2}{5} + \frac{1}{4} = \frac{17}{20}$$

Rm If C is a simple
(does not intersect itself)
closed curve, we use

$\oint_C \vec{F} \cdot \vec{T} ds$ to specify
the orientation of C .

Related integral

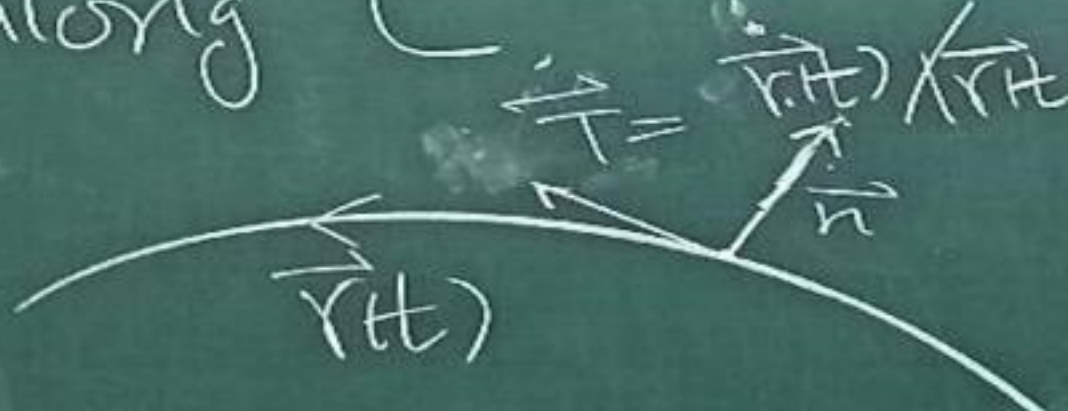
$$(3) \int_C \vec{F} \cdot \vec{n} ds \quad (\text{flux})$$

C : simple closed curve
in a plane

\vec{n} : outward unit normal

To compute the flux, we need to compute \vec{n} from $\vec{r}(t)$. For example, if $\vec{r}(t)$ is counterclockwise

along C

$$\vec{T} = \frac{\vec{r}'(t) \times (\vec{r}'(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$


$$\vec{n} \Rightarrow \begin{aligned} n_1 &= T_2 \\ n_2 &= -T_1 \end{aligned}$$


$$\therefore \vec{n} ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}} \sqrt{\dot{x}^2 + \dot{y}^2} dt = (\dot{y}(t), -\dot{x}(t)) dt$$

Ex. $\vec{F} = (x-y, y)$, $C = \{x^2 + y^2 = 1\}$

$\oint_C \vec{F} \cdot \vec{T} ds = ?$ $\oint_C \vec{F} \cdot \vec{n} ds = ?$
 $(\vec{F} \cdot d\vec{r})$

Sol: $C: x(t) = \cos t, y(t) = \sin t$ $0 \leq t \leq 2\pi$
 $\dot{x}(t) = -\sin t, \dot{y}(t) = \cos t$

(i) $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{(F_1 \cdot \dot{x} + F_2 \cdot \dot{y})}_{F_1 dx + F_2 dy} dt$
 $= \int_0^{2\pi} ((\cos t - \sin t)(-\sin t) + \sin t \cos t) dt$
 $= \int_0^{2\pi} \sin^2 t dt = \pi$

$$(ii) \quad \vec{T} = (-\sin t, \cos t)$$

$$\vec{n} = (\cos t, \sin t)$$



$$d\vec{r} = \vec{T} ds = (dx, dy)$$

$$\vec{n} ds = (dy, -dx)$$

$$\oint_C \vec{F} \cdot \vec{n} ds = \int_0^{2\pi} F_1 dy - F_2 dx$$

$$= \int_0^{2\pi} (F_1 \dot{y} - F_2 \dot{x}) dt$$

$$= \int_0^{2\pi} (\cos t - \sin t) \cos t - \sin t (-\sin t) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t) dt = 2\pi$$

Thm If f is diff. in D

∇f is cont. in D and C is
any curve in D from point A
to point B , then

$$\int_C \underbrace{\nabla f \cdot \vec{T}}_{\nabla f \cdot d\vec{r}} ds = f(B) - f(A)$$

pf: Let $\vec{r}(t), 0 \leq t \leq 1$ be a parametrization
of C , $\vec{r}(0) = A$, $\vec{r}(1) = B$.

$$\begin{aligned} \Rightarrow \int_C \nabla f \cdot \vec{T} ds &= \int_0^1 (f_x(x(t), y(t)) \dot{x} + f_y(x(t), y(t)) \dot{y}) dt \\ &= \int_0^1 \frac{d}{dt} f(x(t), y(t)) dt = f(x(t), y(t)) \Big|_{t=0}^1 = f(B) - f(A) \end{aligned}$$

Fundamental Theorem of line integral

Thm 1 If f is differentiable in D

, ∇f is continuous in D and

C is any curve in D

from point A to point B , then

$$\int_C \nabla f \cdot \vec{T} ds = f(B) - f(A)$$

If Let $C = \{ \vec{r}(t), 0 \leq t \leq 1 \}$

with $A = \vec{r}(0)$, $B = \vec{r}(1)$

$$\begin{aligned} \Rightarrow \int_C \nabla f \cdot \vec{T} ds &= \int_0^1 \left(f_x(x(t), y(t), z(t)) \dot{x}(t) \right. \\ &\quad \left. + f_y(x(t), y(t), z(t)) \dot{y}(t) \right. \\ &\quad \left. + f_z(x(t), y(t), z(t)) \dot{z}(t) \right) dt \\ &= \int_0^1 \frac{d}{dt} f(x(t), y(t), z(t)) dt \\ &= f(x(t), y(t), z(t)) \Big|_{t=0}^1 = f(B) - f(A) \end{aligned}$$

Ex 1 Evaluate $\int_A^B \vec{F} \cdot \vec{T} ds$

along C , where $A = (2, 0)$

$B = (-2, 0)$ $C = \left\{ \frac{x^2}{4} + y^2 = 1, y \geq 0 \right\}$

$$\vec{F} = \frac{(x, y)}{\sqrt{x^2 + y^2}}$$

Sol. $\vec{F} = \nabla(\sqrt{x^2 + y^2})$

$$\Rightarrow \int_A^B \vec{F} \cdot \vec{T} ds$$

$$= \int_A^B \nabla(\sqrt{x^2 + y^2}) \cdot \vec{T} ds$$

$$= \sqrt{x^2 + y^2} \Big|_{\substack{(-2, 0) = B \\ (2, 0) = A}} = 0$$

Remark: If $\vec{F} = \nabla f$

Then $\int_C \vec{F} \cdot \vec{T} ds = 0$

for any closed curve C

Def. D is "connected"

If any two point in D
can be connected by a
path (curve) in D



connected

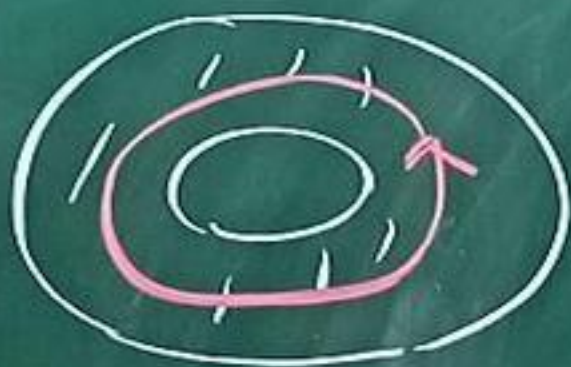


Not connected

Def D is Simply connected

If any closed curve in D
can be continuously shrunk
to a point in D without leaving D

Eg



$$D = \{(x, y), 1 \leq x^2 + y^2 \leq 4\}$$

Not Simply connected



: No

$$D = \mathbb{R}^2 - \{(0,0)\}$$

No

$$D = \{(x, y, z), 1 \leq x^2 + y^2 \leq 4\} : \text{No}$$

$$D = \{(x, y, z), 1 \leq x^2 + y^2 + z^2 \leq 4\}, \text{Yes}$$

$$D = \mathbb{R}^3 - \{(0,0,0)\} : \text{Yes}$$

Def: \vec{F} is conservative

if $\int_C \vec{F} \cdot d\vec{r}$ only depends

on starting point and

end point of C , (and

not on the rest of C).

Thm 1 (assumptions)

$\vec{F} = \nabla f \implies \vec{F}$ is conservative

Thm 2 (If D is connected)

" \Leftarrow " (proof: textbook)

Thm 3

\vec{F} is conservative in D

$\Leftrightarrow \oint_C \vec{F} \cdot d\vec{r}$ for any closed curve $C \subset D$.



Component test (and 2D version)

Let $\vec{F} = (\underbrace{M(x,y,z)}, \underbrace{N(x,y,z)}, P(x,y,z))$
with $\underbrace{M}, \underbrace{N}, \underbrace{P}$ and their partial
derivative continuous in D .

Then

$$(i) \vec{F} \text{ is conservative} \Rightarrow \begin{cases} \underline{M_y = N_x} \\ N_z = P_y \quad (*) \\ P_x = M_z \end{cases}$$

(ii) " \Leftarrow " if D is Simply connected

Question:

When is \vec{F} conservative?

i.e. Can we find the potential f ?

Eg 1. $\vec{F} = (e^x \cos y + yz, xz - e^x \sin y, xy + z)$

Is \vec{F} conservative?

Sol: Check component test first

Is (*) true? No \Rightarrow Not conservative
Yes \Rightarrow try to find f .

$$\begin{aligned} \text{check. } M_y &= -e^x \sin y + z = N_x \\ N_z &= x = P_y \\ P_x &= y = M_z \end{aligned}$$

(*) holds \Rightarrow f may probably exist!

How to find f ?

$$f_x = e^x \cos y + yz$$

$$\Rightarrow f = \int (e^x \cos y + yz) dx$$

$$= e^x \cos y + xyz + C_1(y, z)$$

Similarly

$$f = xyz + e^x \cos y + C_2(x, z)$$

$$f = xyz + \frac{z^2}{2} + C_3(x, y)$$

$$\Rightarrow f(x, y, z)$$

$$= xyz + e^x \cos y + \frac{z^2}{2} \text{ will do!}$$

Ex 2. $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

Is \vec{F} conservative?

Sol: $M_y = N_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\therefore (*)$ holds!

\vec{F} is defined on $D = \mathbb{R}^3 \setminus z\text{-axis}$

D is not simply connected.

Try to find $f \implies$ does not work

Try to prove \vec{F} is not conservative.

\implies Use Thm 2

Take $C: \vec{r}(t) = (a \cos t, a \sin t, 0)$

with $a > 0$ fixed, $0 \leq t \leq 2\pi$

$$\dot{\vec{r}}(t) = (-a \sin t, a \cos t, 0)$$

$$\oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{a}, \frac{\cos t}{a}, 0 \right) \cdot (-a \sin t, a \cos t, 0) dt$$

$$= 2\pi$$

From Thm 2 $\Rightarrow \vec{F}$ is not conservative.

Ex 3: $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right)$ conservative?

Sol: $N_y = M_x = \frac{1}{2} (xy'(x^2+y^2)^{-\frac{3}{2}})$

$\therefore f$ may exist. Take same C as in Ex 2,

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (c, s, 0) \cdot (-as, ac, 0) dt = 0$$

Try to find $f \Rightarrow f(x, y) = \sqrt{x^2+y^2} + C$

$\therefore \vec{F}$ is conservative!