Line integrals

Goal $\displaystyle\int_{\vec{p}}^{\vec{q}} \nabla f(\vec{r})\cdot \underbrace{\vec{\tau}\, ds}_{d\vec{r}} = f(\vec{q}) - f(\vec{p})$

$(\vec{p}, \vec{q} \in \mathbb{R}^3 \quad f: \mathbb{R}^3 \longmapsto \mathbb{R})$

We start with a related integral

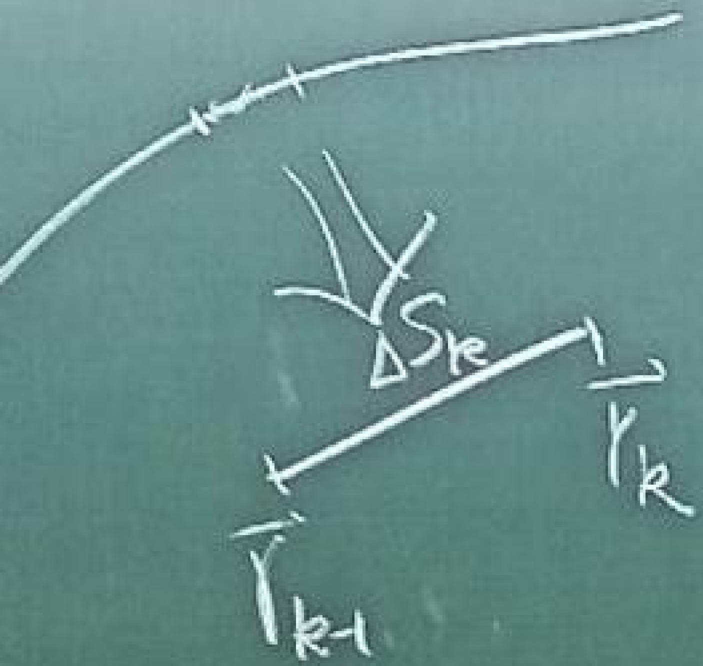
$$\int_C f(x,y,z)\, ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta S_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.

$$C = \left\{ \vec{r}(t),\ a \le t \le b \right\}$$

$$= \left\{ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},\ a \le t \le b \right\}$$



$$\vec{r}_k = \vec{r}(t_k) \qquad \Delta S_k = \left| \vec{r}_k - \vec{r}_{k-1} \right|$$

$$\Delta t_k = t_k - t_{k-1} \qquad = \frac{\left| \vec{r}_k - \vec{r}_{k-1} \right|}{\Delta t_k} \Delta t_k$$

$$\therefore\ dS = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$

Line integrals

Eg 1 Evaluate $\int_C f(x,y,z)\,ds$

where $f = x - 3y^2 + z$

C = line segment between $(0,0,0)$ and $(1,1,1)$.

Sol: Step 1: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t,t,t)$, $0 \le t \le 1$

Step 2: $ds = |\dot{\vec{r}}(t)|\,dt = |(1,1,1)|\,dt$

Step 3: $\text{Ans} = \int_{t=0}^{1} (t - 3t^2 + t)\sqrt{3}\,dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Rm: the value of $I = \int_C f(x,y,z)\,ds$

is independent of the parameter

$\vec{r}(t)$. For example

$\vec{r}(t) = (t^2, t^2, t^2)$, $0 \le t \le 1$

or $\vec{r}(t) = (1-t, 1-t, 1-t)$, $0 \le t \le 1$

all give the same value of

I as long as $\vec{r}(t)$ is

a correct parametrization

for C.

Eg2: $f(x,y,z) = x - 3y^2 + z$

$C = C_1 \cup C_2$ (two line segments)

$C_1 = \overline{(0,0,0)\ (1,1,0)}$

$C_2 = \overline{(1,1,0)\ (1,1,1)}$

<u>Sol</u>: $C_1: \vec{r_1}(t) = (t, t, 0),\ 0 \le t \le 1$

$C_2: \vec{r_2}(t) = (1, 1, t),\ 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$= \int_0^1 (t - 3t^2)\underbrace{\sqrt{2}}_{\left|\frac{d\vec{r_1}}{dt}\right|}\,dt + \int_0^1 (-2+t)\cdot\underbrace{1}_{\left|\frac{d\vec{r_2}}{dt}\right|}\cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2}$$

(Compare with Eg1: same f, different C, (same end points) $\Rightarrow$ different answers)

Related integral

(2) $\int_C \vec{F} \cdot \vec{T}\, ds$

$$\vec{F} : D \longrightarrow \mathbb{R}^3$$
$$(x, y, z) \quad (F_1(x,y,z), F_2(\cdots), F_3(\cdots))$$

C: a smooth curve with prescribed orientation (指向)

$\vec{r}(t)$: parametrization of C with "direction of increasing t" = "orientation of C"

$$\vec{T} = \dot{\vec{r}}(t) / |\dot{\vec{r}}(t)|$$

Eg 3 Evaluate $\int_C \vec{F} \cdot \vec{T}\, ds$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

where $\vec{F} = (z, xy, -y^2)$

$C: \vec{r}(t) = (t^2, t, \sqrt{t}),\ 0 \le t \le 1$

<u>Sol.</u> $\dot{\vec{r}}(t) = \left(2t, 1, \frac{1}{2\sqrt{t}}\right)$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_{t=0}^{1} \vec{F} \cdot \frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|} |\dot{\vec{r}}(t)|\, dt$$

$$= \int_0^1 \underbrace{(\sqrt{t}, t^3, -t^2)}_{\vec{F}} \cdot \underbrace{\left(2t, 1, \frac{1}{2\sqrt{t}}\right)}_{\dot{\vec{r}}(t)} dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2}\right) dt$$

$$= \frac{3}{2} \cdot \frac{2}{5} + \frac{1}{4} = \frac{17}{20}$$

Rm If C is a simple (does not intersect itself) closed curve, we use $\oint_C \vec{F}\cdot\vec{T}\,ds$ to specify the orientation of C.

Related integral

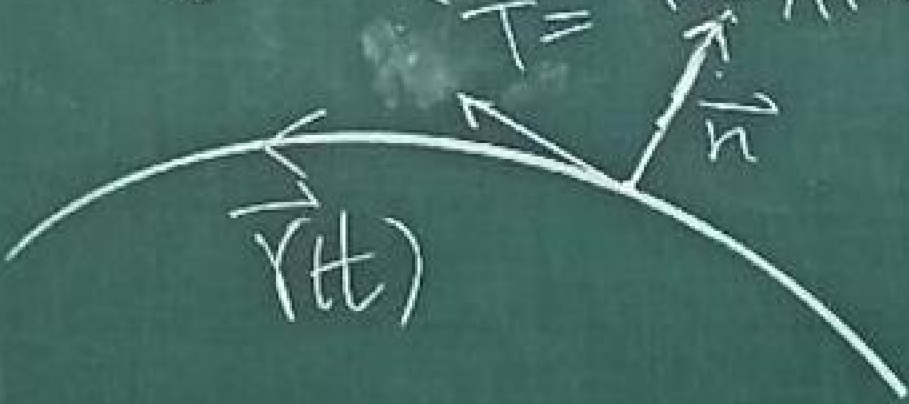
(3) $\int_C \vec{F}\cdot\vec{n}\,ds$ (flux)

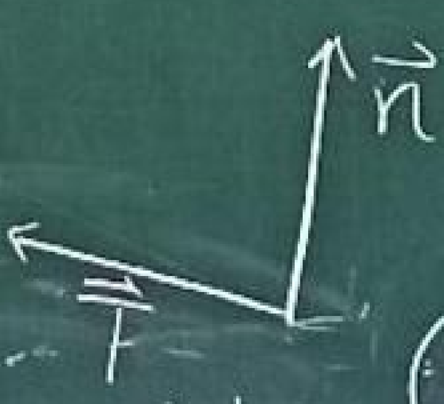
C: simple closed curve in a plane

$\vec{n}$: outward unit normal

To compute the flux, we need to compute $\vec{n}$ from $\vec{r}(t)$. For example, if $\vec{r}(t)$ is counterclockwise along C,

$$\vec{T} = \dot{\vec{r}}(t) / |\dot{\vec{r}}(t)| = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$





$$\Rightarrow \begin{aligned} n_1 &= T_2 \\ n_2 &= -T_1 \end{aligned}$$

$$\therefore \vec{n}\, ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = (\dot{y}(t), -\dot{x}(t))\, dt$$

Eg. $\vec{F} = (x-y, y)$, $C = \{x^2+y^2=1\}$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = ?\,, \quad \oint_C \vec{F}\cdot\vec{n}\,ds = ?\,,$$

$(\vec{F}\cdot d\vec{r})$

Sol: $C: x(t)=\cos t,\ y(t)=\sin t$, $0\le t\le 2\pi$

$\dot{x}(t) = -\sin t,\ \dot{y}(t)=\cos t$

(i) $$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{(F_1\cdot\dot{x} + F_2\cdot\dot{y})dt}_{F_1dx+F_2dy}$$

$$= \int_0^{2\pi} \left((\cos t - \sin t)(-\sin t) + \sin t\cos t\right)dt$$

$$= \int_0^{2\pi} \sin^2 t\,dt = \pi$$

(ii) $\vec{T} = (-\sin t, \cos t)$

$\vec{n} = (\cos t, \sin t)$

$$d\vec{r} = \vec{T}\,ds = (dx, dy)$$

$$\vec{n}\,ds = (dy, -dx)$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} F_1\,dy - F_2\,dx$$

$$= \int_0^{2\pi} \underline{(F_1\dot{y} - F_2\dot{x})\,dt}$$

$$= \int_0^{2\pi} (\cos t - \sin t)\cos t - \sin t(-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi$$

<u>Thm</u> If f is diff. in D

∇f is cont. in D and C is any carve in D from point A to point B, then

$$\int_C \underbrace{\nabla f \cdot \vec{T}\, ds}_{\nabla f \cdot d\vec{r}} = f(B) - f(A)$$

<u>pf</u>: Let $\vec{r}(t)$, $0 \le t \le 1$ be a parametrization of C, $\vec{r}(0) = A$, $\vec{r}(1) = B$.

$$\Rightarrow \int_C \nabla f \cdot \vec{T}\, ds = \int_0^1 \left(f_x(x(t), y(t))\, \dot{x} + f_y(x(t), y(t))\, \dot{y}(t) \right) dt$$

$$= \int_0^1 \frac{d}{dt} f(x(t), y(t))\, dt = f(x(t), y(t)) \Big|_{t=0}^{1} = f(B) - f(A)$$

Fundamental Theorem of line integral

Thm 1 If f is differentiable in D, ∇f is continuous in D and C is <u>any</u> curve in D <u>from</u> point A <u>to</u> point B, then

$$\int_C \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$$

Pf Let $C = \{\vec{r}(t),\ 0 \le t \le 1\}$ with $A = \vec{r}(0)$, $B = \vec{r}(1)$

$$\Rightarrow \int_C \nabla f \cdot \vec{T}\, ds = \int_0^1 \begin{pmatrix} f_x(x(t), y(t), z(t))\, \dot{x}(t) \\ + f_y(x(t), y(t), z(t))\, \dot{y}(t) \\ + f_z(x(t), y(t), z(t))\, \dot{z}(t) \end{pmatrix} dt$$

$$= \int_0^1 \frac{d}{dt} f(x(t), y(t), z(t))\, dt$$

$$= f(x(t), y(t), z(t)) \Big|_{t=0}^{1} = f(B) - f(A)$$

Eg: Evaluate $\int_A^B \vec{F}\cdot\vec{T}\,ds$

along C, where $A=(2,0)$

$B=(-2,0)$ $C=\{\frac{x^2}{4}+y^2=1,\ y\geq 0\}$

$$\vec{F}=\frac{(x,y)}{\sqrt{x^2+y^2}}$$

<u>Sol</u>: $\vec{F}=\nabla\left(\sqrt{x^2+y^2}\right)$

$$\Rightarrow \int_A^B \vec{F}\cdot\vec{T}\,ds$$

$$=\int_A^B \nabla\left(\sqrt{x^2+y^2}\right)\cdot\vec{T}\,ds$$

$$=\sqrt{x^2+y^2}\,\Big|_{\underset{A}{(2,0)}}^{(-2,0)=B}=0$$

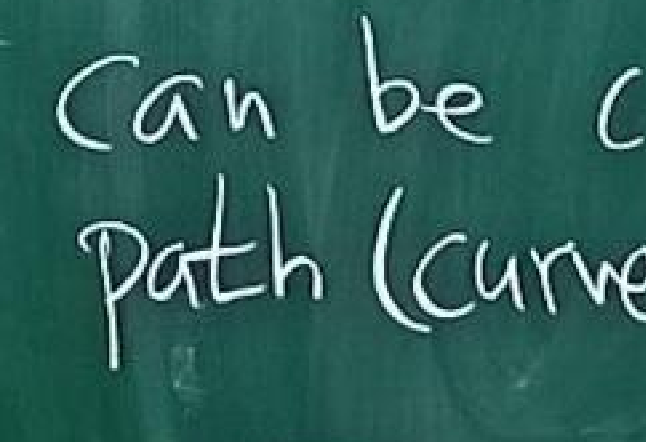
Remark: If $\vec{F} = \nabla f$

Then $\int_C \vec{F} \cdot \vec{T} \, ds = 0$

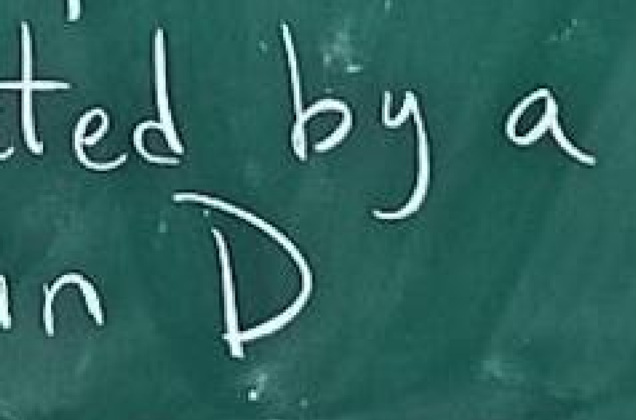
for any closed curve C

Def. D is "connected"

If any two point in D can be connected by a path (curve) in D



Connected



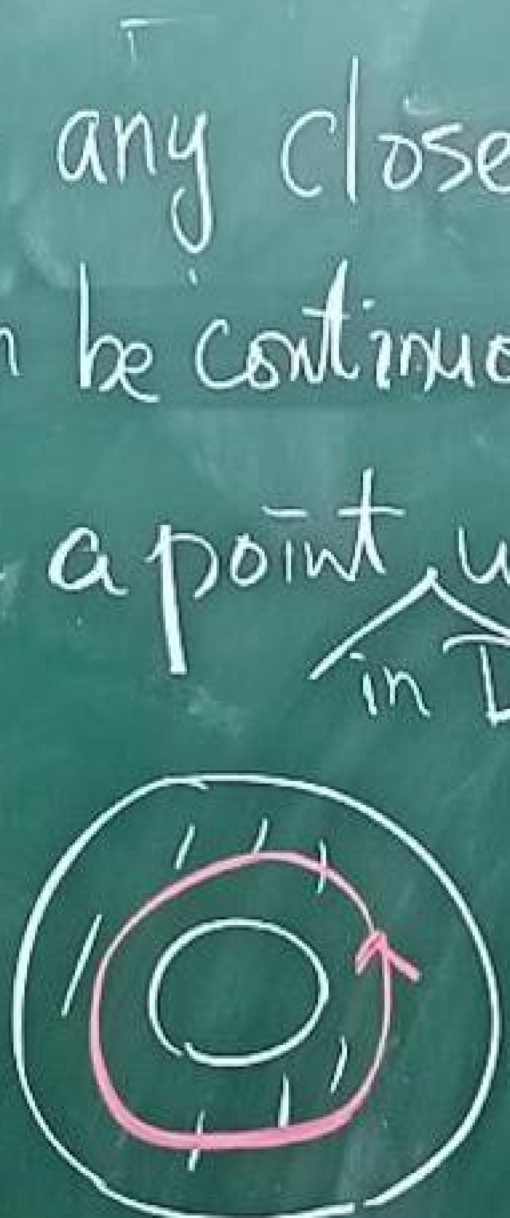
Not connected

Def. D is simply connected

If any closed curve in D can be continuously shrinked to a point without leaving D

in D

Eg



$D = \{(x,y),\ 1 \le x^2+y^2 \le 4\}$

Not simply connected

: No

$\boxed{D = \mathbb{R}^2 \setminus \{(0,0)\}}$ No

$D = \{(x,y,z)\ 1 \le x^2+y^2 \le 4\}$: No

$D = \{(x,y,z),\ 1 \le x^2+y^2+z^2 \le 4\}$, Yes

$D = \mathbb{R}^3 \setminus \{(0,0,0)\}$: Yes

Def: $\vec{F}$ is conservative if $\int_C \vec{F} \cdot d\vec{r}$ only depends on starting point and end point of C. (and not on the rest of C).

Thm 1 (assumptions)

$\vec{F} = \nabla f \Longrightarrow \vec{F}$ is conservative.

Thm 2 (If D is connected)

" $\Longleftarrow$ " (proof: textbook)

Thm 3

$\vec{F}$ is conservative in D

$\Leftrightarrow \oint_C \vec{F}\cdot d\vec{r}$ for any closed curve $C \subseteq D$.

Component test (and 2D version)

Let $\vec{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$

with M, N, P and their partial derivative continuous in D.

Then

(i) $\vec{F}$ is conservative $\Rightarrow \begin{cases} M_y = N_x \\ N_z = P_y \\ P_x = M_z \end{cases}$ (*)

(ii) "$\Leftarrow$" if D is simply connected

Question:

When is $\vec{F}$ conservative?

i.e. Can we find the potential f?

Eg1. $\vec{F} = (e^x \cos y + yz,\ xz - e^x \sin y,\ xy + z)$

Is $\vec{F}$ conservative?

Sol: Check component test first

Is $(*)$ true? No $\Longrightarrow$ Not conservative

Yes $\Rightarrow$ try to find f.

check. $M_y = -e^x \sin y + z = N_x$

$N_z = x = P_y$

$P_x = y = M_z$

(*) holds $\Rightarrow f$ may probably exist!

How to find f?

$$f_x = e^x \cos y + yz$$

$$\Rightarrow f = \int (e^x \cos y + yz)\,dx$$

$$= e^x \cos y + xyz + C_1(y,z)$$

Similarly

$$f = xyz + e^x \cos y + C_2(x,z)$$

$$f = xyz + \frac{z^2}{2} + C_3(x,y)$$

$\Rightarrow f(x,y,z)$

$= xyz + e^x \cos y + \frac{z^2}{2}$ will do!

Eg2. $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

Is $\vec{F}$ conservative?

<u>Sol</u>: $M_y = N_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\therefore$ (*) holds!

$\vec{F}$ is defined on $D = \mathbb{R}^3 \setminus z$ axis

D is <u>not</u> simply connected.

Try to find $f \Longrightarrow$ does not work

Try to prove $\vec{F}$ is not conservative.

$\Rightarrow$ Use Thm 2

Take C: $\vec{r}(t) = (a\cos t, a\sin t, 0)$

with $a > 0$ fixed, $0 \leq t \leq 2\pi$

$\dot{\vec{r}}(t) = (-a\sin t, a\cos t, 0)$

$$\oint_C \vec{F}\cdot d\vec{r}$$
$$= \int_0^{2\pi} \left(\frac{-\sin t}{a}, \frac{\cos t}{a}, 0\right)\cdot(-a\sin t, a\cos t, 0)\,dt$$
$$= 2\pi$$

From Thm 2 $\Rightarrow \vec{F}$ is not conservative.

Eg 3: $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$ conservative?

Sol: $N_y = M_x = \frac{-1}{2}\left(xy(x^2+y^2)^{\frac{-3}{2}}\right)$

$\therefore f$ may exist. Take same C as in Eg 2.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (c,s,0)\cdot(-as, ac, 0)\,dt = 0$$

Try to find $f \Rightarrow f(x,y) = \sqrt{x^2+y^2} + C$

$\therefore \vec{F}$ is conservative!