

Line integrals

Goal $\int_{\vec{P}}^{\vec{Q}} \nabla f(\vec{r}) \cdot \underbrace{\vec{T}}_{d\vec{r}} ds = f(\vec{Q}) - f(\vec{P})$

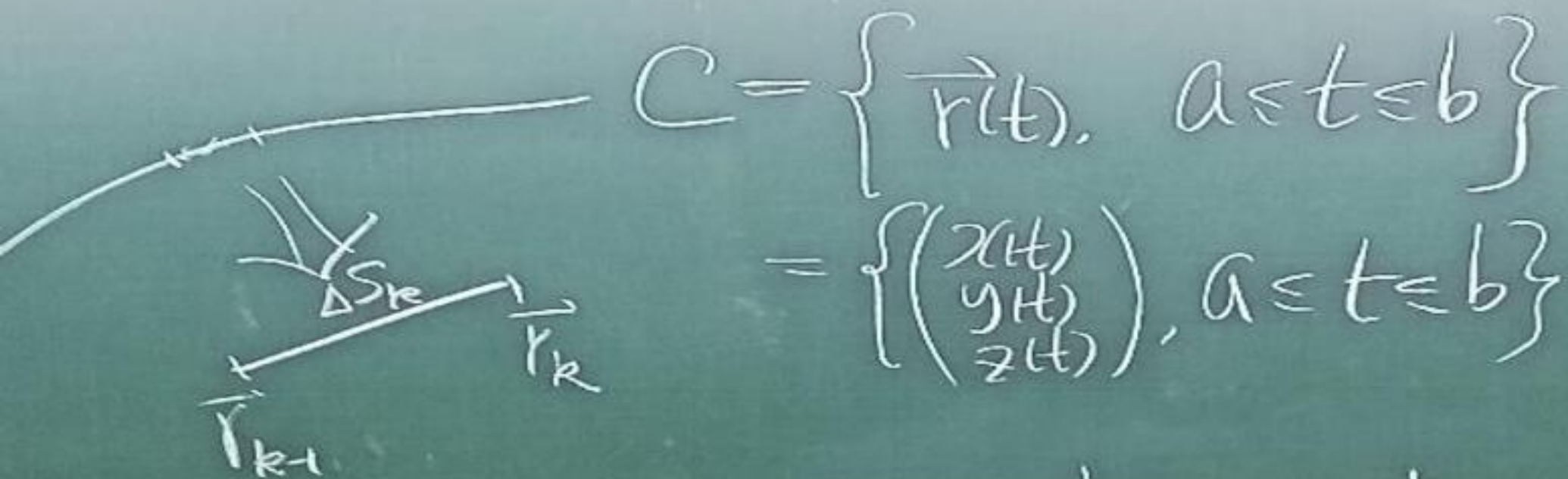
$(\vec{P}, \vec{Q} \in \mathbb{R}^3 \quad f: \mathbb{R}^3 \rightarrow \mathbb{R})$

We start with a related integral

$$\int_C f(x, y, z) ds = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta S_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.



$$\vec{r}_k = \vec{r}(t_k)$$

$$\Delta t_k = t_k - t_{k-1}$$

$$\Delta S_k = |\vec{r}_k - \vec{r}_{k-1}|$$

$$= \frac{|\vec{r}_k - \vec{r}_{k-1}|}{\Delta t_k} \Delta t_k$$

$$\therefore dS = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$

Line integrals

Eg 1. Evaluate $\int_C f(x, y, z) ds$

where $f = x - 3y^2 + z$

C = line segment between
 $(0, 0, 0)$ and $(1, 1, 1)$.

Sol: Step 1: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t, t, t)$, $0 \leq t \leq 1$

Step 2: $ds = |\vec{r}'(t)| dt = |(1, 1, 1)| dt$

Step 3: $\text{Ans} = \int_{t=0}^1 (t - 3t^2 + t) \sqrt{3} dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

Rm: the value of $I = \int_C f(x, y, z) ds$,
is independent of the parameter
 $\vec{r}(t)$. For example

$$\vec{r}(t) = (t^2, t^2, t^2), \quad 0 \leq t \leq 1$$

$$\text{or } \vec{r}(t) = (1-t, 1-t, 1-t), \quad 0 \leq t \leq 1$$

all give the same value of

I as long as $\vec{r}(t)$ is
a correct parametrization
for C .

$$\text{Eq 2: } f(x, y, z) = x - 3y^2 + z$$

$$C = C_1 \cup C_2 \quad (\text{two line segments})$$

$$C_1 = \overline{(0, 0, 0) \quad (1, 1, 0)}$$

$$C_2 = \overline{(1, 1, 0) \quad (1, 1, 1)}$$

Sol: $C_1: \vec{r}_1(t) = (t, t, 0), 0 \leq t \leq 1$

$$C_2: \vec{r}_2(t) = (1, 1, t), 0 \leq t \leq 1$$

$$S_C = S_{C_1} + S_{C_2} \quad \left| \frac{d\vec{r}_1}{dt} \right| \quad \left| \frac{d\vec{r}_2}{dt} \right|$$

$$= \int_0^1 (t - 3t^2) \sqrt{2} dt + \int_0^1 (-2 + t) \cdot 1 \cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2}$$

(Compare with Eq 1; same f
different C , (same end points)
 \rightarrow different answers)

Related integral

$$(2) \int_C \vec{F} \cdot \vec{T} \, ds$$

$$\vec{F}: D \longrightarrow \mathbb{R}^3$$

$$(x, y, z) \quad (F_1(x, y, z), F_2(\cdot), F_3(\cdot))$$

C : a smooth curve **in D**
with prescribed orientation (指向)

$\vec{r}(t)$: parametrization of C with

"direction of increasing t "

= "orientation of C "

$$\vec{T} = \frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|} \Rightarrow \vec{T} \, ds = \dot{\vec{r}}(t) \, dt$$

Eg 3 Evaluate $\int_C \vec{F} \cdot \vec{T} ds$

where $\vec{F} = (z, xy, -y^2)$

$$= \int_C \vec{F} \cdot d\vec{F} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$C: \vec{r}(t) = (t^2, t, \sqrt{t}), 0 \leq t \leq 1$

Sol. $\vec{r}(t) = (2t, 1, \frac{1}{2\sqrt{t}})$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_{t=0}^1 \vec{F} \cdot \frac{\vec{r}(t)}{|\vec{r}(t)|} |\dot{\vec{r}}(t)| dt$$

$$= \int_0^1 (\sqrt{t}, t^3, -t^2) \cdot (2t, 1, \frac{1}{2\sqrt{t}}) dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2} \right) dt$$

$$= \frac{3}{2} \cdot \frac{2}{5} + \frac{1}{4} = \frac{17}{20}$$

Rm If C is a simple
(does not intersect itself)
closed curve, we use

$\oint_C \vec{F} \cdot \vec{T} ds$ to specify
the orientation of C .

Related integral

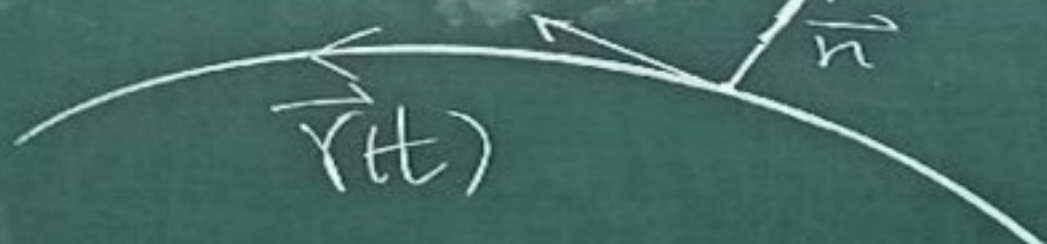
$$(3) \int_C \vec{F} \cdot \vec{n} ds \quad (\text{flux})$$

C : simple closed curve
in a plane

\vec{n} : outward unit normal

To compute the flux, we need to compute \vec{n} from $\vec{r}(t)$. For example, if $\vec{r}(t)$ is counterclockwise

along C $\vec{T} = \frac{d\vec{r}(t)}{dt} = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$



$\vec{n} \Rightarrow \underline{n_1 = T_2}$
 $\underline{n_2 = -T_1}$

$\therefore \vec{n} ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}} \sqrt{\dot{x}^2 + \dot{y}^2} dt = (\dot{y}(t), -\dot{x}(t)) dt$

$$\text{Eg 4: } \vec{F} = (x-y, y), \quad C = \{x^2 + y^2 = 1\}$$

$$\oint_C \vec{F} \cdot \vec{T} ds = ? \quad \oint_C \vec{F} \cdot \vec{n} ds = ?$$

$(\vec{F} \cdot d\vec{r})$

Sol: $C: x(t) = \cos t, y(t) = \sin t$
 $\dot{x}(t) = -\sin t, \dot{y}(t) = \cos t$ $0 \leq t \leq 2\pi$

$$\begin{aligned} \text{(i) } \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \underbrace{(F_1 \cdot \dot{x} + F_2 \cdot \dot{y})}_{F_1 dx + F_2 dy} dt \\ &= \int_0^{2\pi} ((\cos t - \sin t)(-\sin t) + \sin t \cos t) dt \\ &= \int_0^{2\pi} \sin^2 t dt = \pi \end{aligned}$$

$$(ii) \quad \vec{T} = (-\sin t, \cos t)$$

$$\vec{n} = (\cos t, \sin t)$$



$$d\vec{r} = \vec{T} ds = (dx, dy)$$

$$\vec{n} ds = (dy, -dx)$$

$$\oint_C \vec{F} \cdot \vec{n} ds = \int_0^{2\pi} F_1 dy - F_2 dx$$

$$= \int_0^{2\pi} (F_1 \dot{y} - F_2 \dot{x}) dt$$

$$= \int_0^{2\pi} (\cos t - \sin t) \cos t - \sin t (-\sin t) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t) dt = 2\pi$$