Line integrals

Goal $\displaystyle\int_{\vec{p}}^{\vec{q}} \nabla f(\vec{r})\cdot \underbrace{\vec{T}\,ds}_{d\vec{r}} = f(\vec{q}) - f(\vec{p})$

$(\vec{p}, \vec{q} \in \mathbb{R}^3 \quad f: \mathbb{R}^3 \longrightarrow \mathbb{R})$

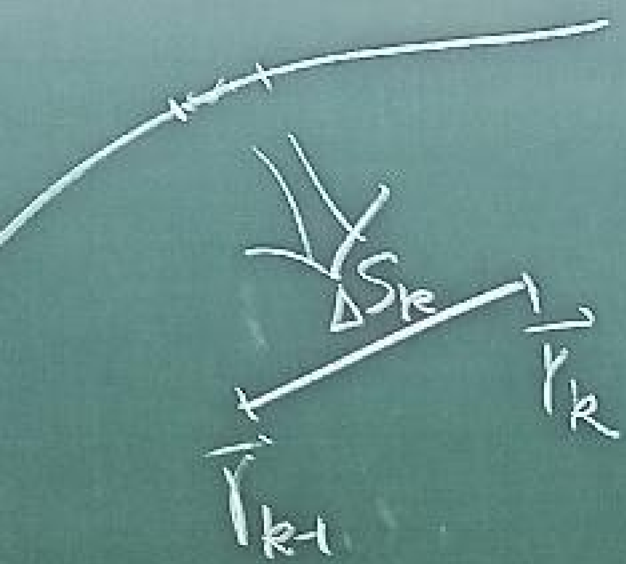
We start with a related integral

$$\int_C f(x,y,z)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.

$$C = \left\{ \vec{r}(t),\ a \le t \le b \right\}$$



$$= \left\{ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},\ a \le t \le b \right\}$$

$$\vec{r}_k = \vec{r}(t_k) \qquad \Delta S_k = \left| \vec{r}_k - \vec{r}_{k-1} \right|$$

$$\Delta t_k = t_k - t_{k-1} \qquad = \frac{\left| \vec{r}_k - \vec{r}_{k-1} \right|}{\Delta t_k} \Delta t_k$$

$$\therefore\ ds = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$

Line integrals

Eg 1 Evaluate $\int_C f(x,y,z)\,ds$

where $f = x - 3y^2 + z$

C = line segment between $(0,0,0)$ and $(1,1,1)$.

Sol: Step 1: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t,t,t),\ 0 \le t \le 1$

Step 2: $ds = |\dot{\vec{r}}(t)|\,dt = |(1,1,1)|\,dt$

Step 3: $Ans = \int_{t=0}^{1} (t - 3t^2 + t)\sqrt{3}\,dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Rm: the value of $I = \int_C f(x,y,z)\,ds$ is independent of the parameter $\vec{r}(t)$. For example

$\vec{r}(t) = (t^2, t^2, t^2)$, $0 \le t \le 1$

or $\vec{r}(t) = (1-t, 1-t, 1-t)$, $0 \le t \le 1$

all give the same value of I as long as $\vec{r}(t)$ is a correct parametrization for C.

Eg2: $f(x,y,z) = x - 3y^2 + z$

$C = C_1 \cup C_2$ (two line segments)

$C_1 = \overline{(0,0,0)\ (1,1,0)}$

$C_2 = \overline{(1,1,0)\ (1,1,1)}$

Sol: $C_1: \vec{r_1}(t) = (t,t,0),\ 0 \le t \le 1$

$C_2: \vec{r_2}(t) = (1,1,t),\ 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$= \int_0^1 (t - 3t^2) \underbrace{\sqrt{2}}_{\left|\frac{d\vec{r_1}}{dt}\right|} dt + \int_0^1 (-2+t) \cdot \underbrace{1}_{\left|\frac{d\vec{r_2}}{dt}\right|} \cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2}$$

(Compare with Eg1: same f, different C (same end points) ⇒ different answers)

Related integral

(2) $\int_C \vec{F}\cdot\vec{T}\,ds$

$\vec{F}: D \longmapsto \mathbb{R}^3$

$(x,y,z) \quad (F_1(x,y,z), F_2(\cdot\cdot), F_3(\cdot\cdot))$

C: a smooth curve in D

with prescribed orientation(指向)

$\vec{r}(t)$: parametrization of C with

"direction of increasing t"

= "orientation of C"

$\vec{T} = \dot{\vec{r}}(t) / |\dot{\vec{r}}(t)| \Rightarrow \vec{T}\,ds = \dot{\vec{r}}(t)\,dt$

Eg3 Evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\,dx+F_2\,dy+F_3\,dz)$$

where $\vec{F}=(z,xy,-y^2)$

$C: \vec{r}(t)=(t^2,t,\sqrt{t}),\ 0\le t\le 1$

Sol. $\dot{\vec{r}}(t)=\left(2t,1,\frac{1}{2\sqrt{t}}\right)$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=0}^{1} \vec{F}\cdot\frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|}\,|\dot{\vec{r}}(t)|\,dt$$

$$=\int_0^1 \underbrace{(\sqrt{t},t^3,-t^2)}_{\vec{F}}\cdot\underbrace{\left(2t,1,\frac{1}{2\sqrt{t}}\right)}_{\dot{\vec{r}}(t)}\,dt$$

$$=\int_0^1\left(2t^{\frac{3}{2}}+t^3-\frac{t^{\frac{3}{2}}}{2}\right)dt$$

$$=\frac{3}{2}\cdot\frac{2}{5}+\frac{1}{4}=\frac{17}{20}$$

Rm If C is a simple (does not intersect itself) closed curve, we use

$$\oint_C \vec{F}\cdot\vec{T}\,ds$$ to specify the orientation of C.

Related integral

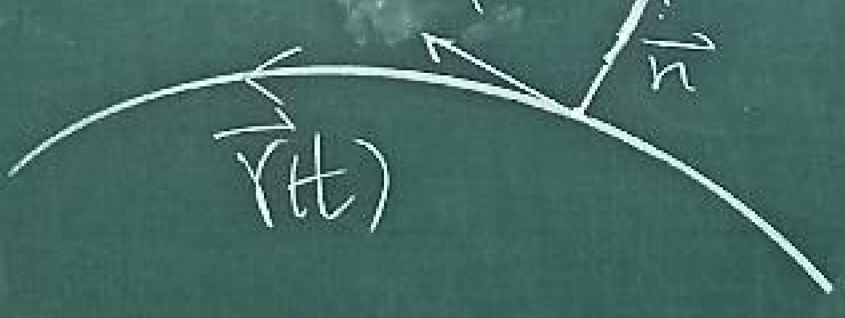
(3) $\displaystyle\int_C \vec{F}\cdot\vec{n}\,ds$ (flux)

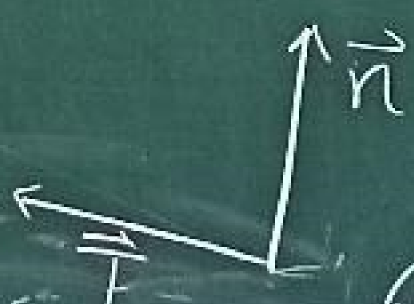
C: simple closed curve in a plane

$\vec{n}$: outward unit normal

To compute the flux, we need to compute $\vec{n}$ from $\vec{r}(t)$. For example, if $\vec{r}(t)$ is counterclockwise along C,

$$\vec{T} = \dot{\vec{r}}(t)/|\dot{\vec{r}}(t)| = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$





$$\Rightarrow \quad n_1 = T_2, \quad n_2 = -T_1$$

$$\therefore \vec{n}\,ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}}\sqrt{\dot{x}^2 + \dot{y}^2}\,dt = (\dot{y}(t), -\dot{x}(t))\,dt$$

Eg4: $\vec{F}=(x-y,y)$, $C=\{x^2+y^2=1\}$

$\oint_C \vec{F}\cdot\vec{T}\,ds=?$ $\quad \oint_C \vec{F}\cdot\vec{n}\,ds=?$

$(\vec{F}\cdot d\vec{r})$

<u>Sol</u>: $C: x(t)=\cos t,\ y(t)=\sin t$, $0\le t\le 2\pi$

$\dot{x}(t)=-\sin t,\ \dot{y}(t)=\cos t$

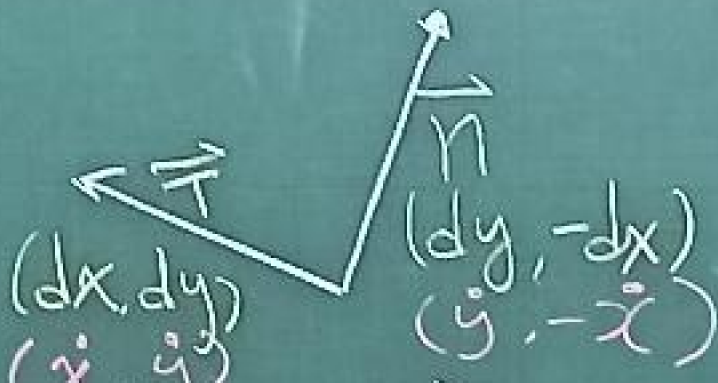
(i) $\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\underbrace{(F_1\cdot\dot{x}+F_2\cdot\dot{y})dt}_{F_1dx+F_2dy}$

$=\int_0^{2\pi}\big((\cos t-\sin t)(-\sin t)+\sin t\cos t\big)dt$

$=\int_0^{2\pi}\sin^2 t\,dt=\pi$

(ii) $\vec{T} = (-\sin t, \cos t)$

$\vec{n} = (\cos t, \sin t)$



$$d\vec{r} = \vec{T}\,ds = (dx, dy)$$

$$\vec{n}\,ds = (dy, -dx)$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} F_1\,dy - F_2\,dx$$

$$= \int_0^{2\pi} \underline{(F_1\dot{y} - F_2\dot{x})\,dt}$$

$$= \int_0^{2\pi} (\cos t - \sin t)\cos t - \sin t(-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi$$