Line integrals

Goal $$\int_{\vec{p}}^{\vec{q}} \nabla f(\vec{r}) \cdot \underbrace{\vec{T}\,ds}_{d\vec{r}} = f(\vec{q}) - f(\vec{p})$$

$(\vec{p}, \vec{q} \in \mathbb{R}^3 \quad f: \mathbb{R}^3 \longrightarrow \mathbb{R})$

We start with a related integral

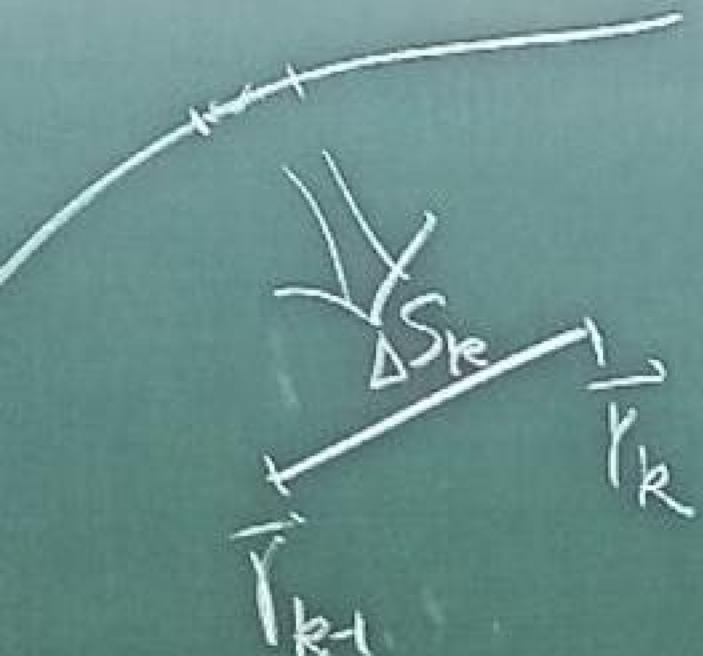
$$\int_C f(x,y,z)\,ds = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.

$$C = \left\{ \vec{r}(t),\ a \le t \le b \right\}$$

$$= \left\{ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},\ a \le t \le b \right\}$$



$$\vec{r}_k = \vec{r}(t_k) \qquad \Delta S_k = \left| \vec{r}_k - \vec{r}_{k-1} \right|$$

$$\Delta t_k = t_k - t_{k-1} \qquad = \frac{\left| \vec{r}_k - \vec{r}_{k-1} \right|}{\Delta t_k} \Delta t_k$$

$$\therefore\ dS = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$

Line integrals

Eg 1 Evaluate $\int_C f(x,y,z)\,ds$

where $f = x - 3y^2 + z$

C = line segment between $(0,0,0)$ and $(1,1,1)$.

Sol: Step 1: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t,t,t)$, $0 \le t \le 1$

Step 2: $ds = |\dot{\vec{r}}(t)|\,dt = |(1,1,1)|\,dt$

Step 3: $\text{Ans} = \int_{t=0}^{1} (t - 3t^2 + t)\sqrt{3}\,dt = 0$

Rm: $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Rm: the value of $I = \int_C f(x,y,z)\,ds$

is independent of the parameter

$\vec{r}(t)$. For example

$\vec{r}(t) = (t^2, t^2, t^2)$, $0 \le t \le 1$

or $\vec{r}(t) = (1-t, 1-t, 1-t)$, $0 \le t \le 1$

all give the same value of

I as long as $\vec{r}(t)$ is

a correct parametrization

for C.

Eg2: $f(x,y,z) = x - 3y^2 + z$

$C = C_1 \cup C_2$ (two line segments)

$C_1 = \overline{(0,0,0) \quad (1,1,0)}$

$C_2 = \overline{(1,1,0) \quad (1,1,1)}$

<u>Sol</u>: $C_1 : \vec{r_1}(t) = (t, t, 0),\ 0 \le t \le 1$

$C_2 : \vec{r_2}(t) = (1, 1, t),\ 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$= \int_0^1 (t - 3t^2)\overset{\left|\frac{d\vec{r_1}}{dt}\right|}{\sqrt{2}}\,dt + \int_0^1 (-2+t)\cdot \overset{\left|\frac{d\vec{r_2}}{dt}\right|}{1} \cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2}$$

(Compare with Eg1: same f, different C, (same end points) $\Rightarrow$ different answers)

Related integral

(2) $\int_C \vec{F} \cdot \vec{T} \, ds$

$\vec{F}: D \longrightarrow \mathbb{R}^3$

$(x, y, z) \quad (F_1(x,y,z), F_2(\cdot\cdot), F_3(\cdot\cdot))$

C: a smooth curve in D

with prescribed orientation (指向)

$\vec{r}(t)$: parametrization of C with

"direction of increasing t"

= "orientation of C"

$\vec{T} = \dot{\vec{r}}(t) / |\dot{\vec{r}}(t)| \Rightarrow \vec{T} ds = \dot{\vec{r}}(t) dt$

Eg3 Evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

where $\vec{F} = (z, xy, -y^2)$

$C : \vec{r}(t) = (t^2, t, \sqrt{t}),\ 0 \le t \le 1$

Sol. $\dot{\vec{r}}(t) = \left(2t, 1, \frac{1}{2\sqrt{t}}\right)$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=0}^{1} \vec{F}\cdot\frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|}\,|\dot{\vec{r}}(t)|\,dt$$

$$= \int_0^1 \underset{\vec{F}}{(\sqrt{t}, t^3, -t^2)}\cdot\underset{\dot{\vec{r}}(t)}{\left(2t, 1, \frac{1}{2\sqrt{t}}\right)}\,dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2}\right)dt$$

$$= \frac{3}{2}\cdot\frac{2}{5} + \frac{1}{4} = \frac{17}{20}$$

Rm If C is a simple (does not intersect itself) closed curve, we use

$$\oint_C \vec{F}\cdot\vec{T}\,ds$$ to specify the orientation of C.

Related integral

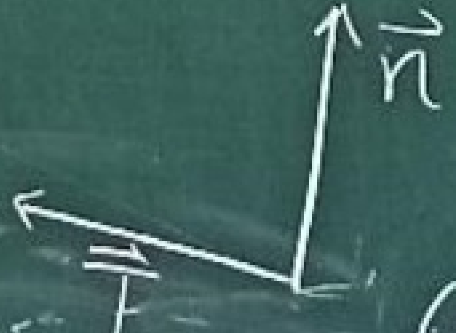
(3) $\int_C \vec{F}\cdot\vec{n}\,ds$ (flux)

C: simple closed curve in a plane

$\vec{n}$: outward unit normal

To compute the flux, we need to compute $\vec{n}$ from $\vec{r}(t)$. For example, if $\vec{r}(t)$ is counterclockwise along C,

$$\vec{T} = \dot{\vec{r}}(t)/|\dot{\vec{r}}(t)| = \frac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$



$$\Rightarrow \underline{n_1 = T_2}, \quad \underline{n_2 = -T_1}$$

$$\therefore \vec{n}\,ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = (\dot{y}(t), -\dot{x}(t))\,dt$$

Eg4: $\vec{F}=(x-y,y)$, $C=\{x^2+y^2=1\}$

$$\oint_C \vec{F}\cdot\vec{T}\,ds=? \qquad \oint_C \vec{F}\cdot\vec{n}\,ds=?$$

$$(\vec{F}\cdot d\vec{r})$$

<u>Sol</u>: $C: x(t)=\cos t,\ y(t)=\sin t \quad 0\le t\le 2\pi$

$\dot{x}(t)=-\sin t,\ \dot{y}(t)=\cos t$

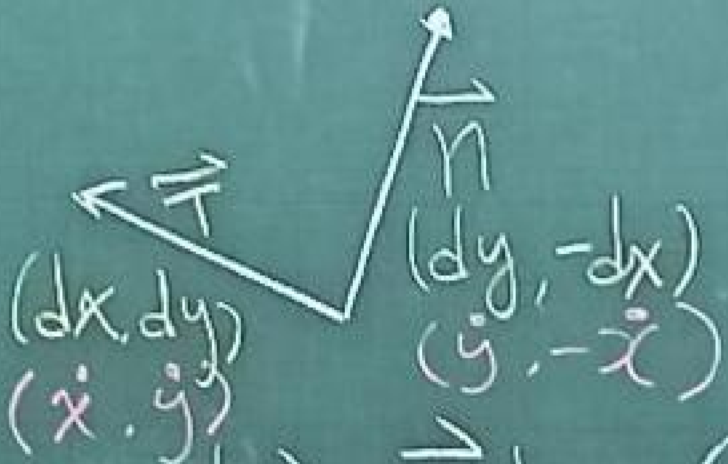
(i) $$\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\underbrace{(F_1\cdot\dot{x}+F_2\cdot\dot{y})dt}_{F_1dx+F_2dy}$$

$$=\int_0^{2\pi}\big((\cos t-\sin t)(-\sin t)+\sin t\cos t\big)dt$$

$$=\int_0^{2\pi}\sin^2 t\,dt=\pi$$

(ii) $\vec{T} = (-\sin t, \cos t)$

$\vec{n} = (\cos t, \sin t)$



$$d\vec{r} = \vec{T}\,ds = (dx, dy)$$

$$\vec{n}\,ds = (dy, -dx)$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} F_1\,dy - F_2\,dx$$

$$= \int_0^{2\pi} \underline{(F_1 \dot{y} - F_2 \dot{x})\,dt}$$

$$= \int_0^{2\pi} (\cos t - \sin t)\cos t - \sin t(-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi$$