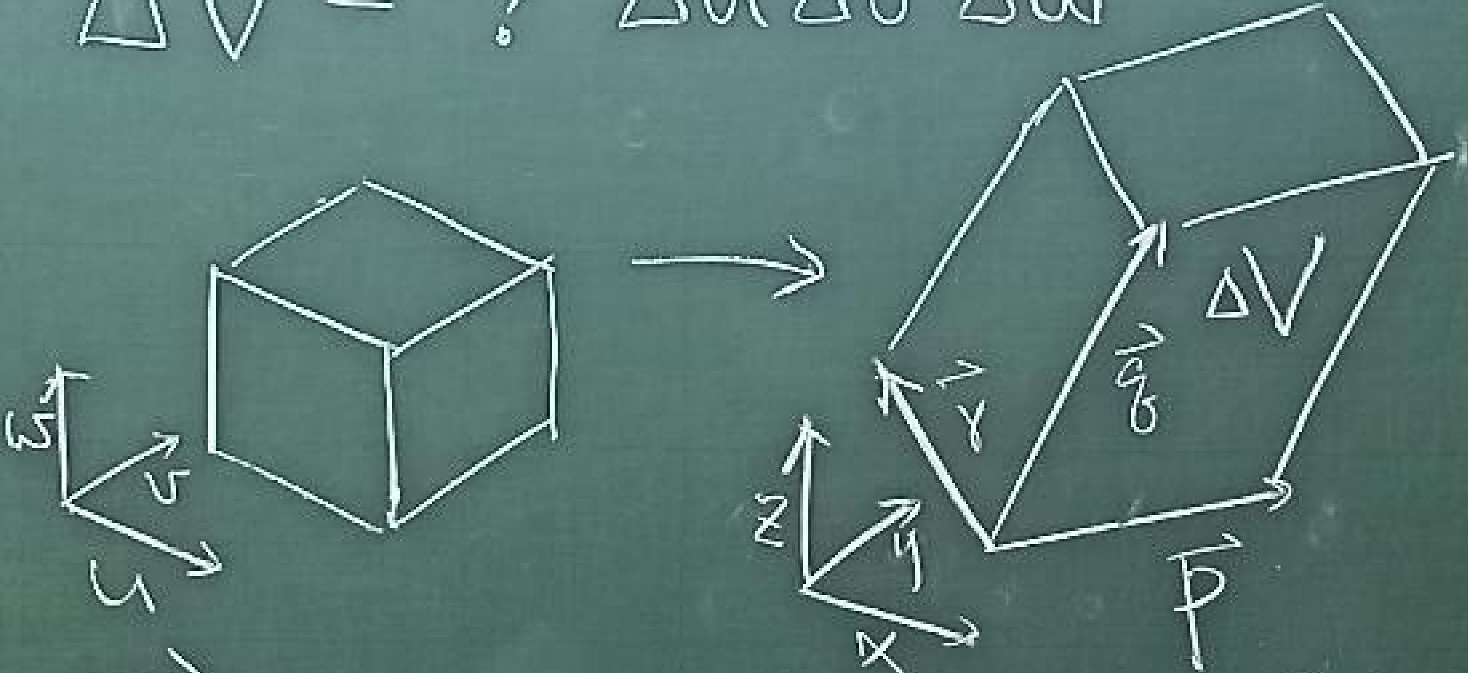
For Triple Integrals

$u(x,y,z),\ v(x,y,z),\ w(x,y,z)$

$\Delta V = ?\ \Delta u \Delta v \Delta w$



$$\vec{p} = \mathbb{X}(u+\Delta u, v, w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_u \Delta u$$

$$\vec{q} = \mathbb{X}(u, v+\Delta v, w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_v \Delta v$$

$$\vec{r} = \mathbb{X}(u, v, w+\Delta w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_w \Delta w$$

$$\Delta V \approx \left| \vec{p} \times \vec{q} \cdot \vec{r} \right| = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right| \Delta u \Delta v \Delta w$$

$$= |J| \Delta u \Delta v \Delta w$$

Summary

Step 1 Find suitable
$u(x,y,z), v(x,y,z), w(x,y,z)$
for the problem.

Step 2 Solve for $x(u,v,w)$
$y(u,v,w), z(u,v,w)$

Step 3.
$$J = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

$$\Delta V = |J| \Delta u \Delta v \Delta w$$

Eg2:

Evaluate $\displaystyle\int_{z=0}^{3}\int_{y=0}^{4}\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$

using $u=x-\frac{y}{2}$, $v=\frac{y}{2}$, $w=\frac{z}{3}$

Sol

$x=u+v$, $y=2v$, $z=3w$

$dw: \displaystyle\int_{z=0}^{3}=\int_{w=0}^{1}$, $dv: \displaystyle\int_{y=0}^{4}=\int_{v=0}^{2}$

$du: \displaystyle\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}$ $\quad x=\frac{y}{2}\Leftrightarrow u=0$

$x=\frac{y}{2}+1\Leftrightarrow u=1$

$$J=\det\begin{pmatrix}x_u & x_v & x_w\\ y_u & y_v & y_w\\ z_u & z_v & z_w\end{pmatrix}=\det\begin{pmatrix}1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$$

$=6$

$$I = \int_{w=0}^{1}\int_{v=0}^{2}\int_{u=0}^{1} (u+w)\,\underline{6\,du\,dv\,dw}$$

$$= 6\left(\iiint u\,du\,dv\,dw + \iiint w\,du\,dv\,dw\right)$$

$$= 6\left(\int_{w=0}^{1}\int_{v=0}^{2} \frac{u^2}{2}\Big|_0^1\,dv\,dw + \int_{v=0}^{2}\int_{u=0}^{1} \left(\frac{w^2}{2}\right)\Big|_0^1\,du\,dv\right)$$

$$= 6\left(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 2\right)$$

$$= 12$$

Eg3: Spherical Coordinate

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$J = \det\begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix}$$

$$= \det\begin{pmatrix} S_\phi C_\theta & \rho C_\phi C_\theta & -\rho S_\phi S_\theta \\ S_\phi S_\theta & \rho C_\phi S_\theta & \rho S_\phi C_\theta \\ C_\phi & -\rho S_\phi & 0 \end{pmatrix} = \rho^2\sin\phi$$

(check!)