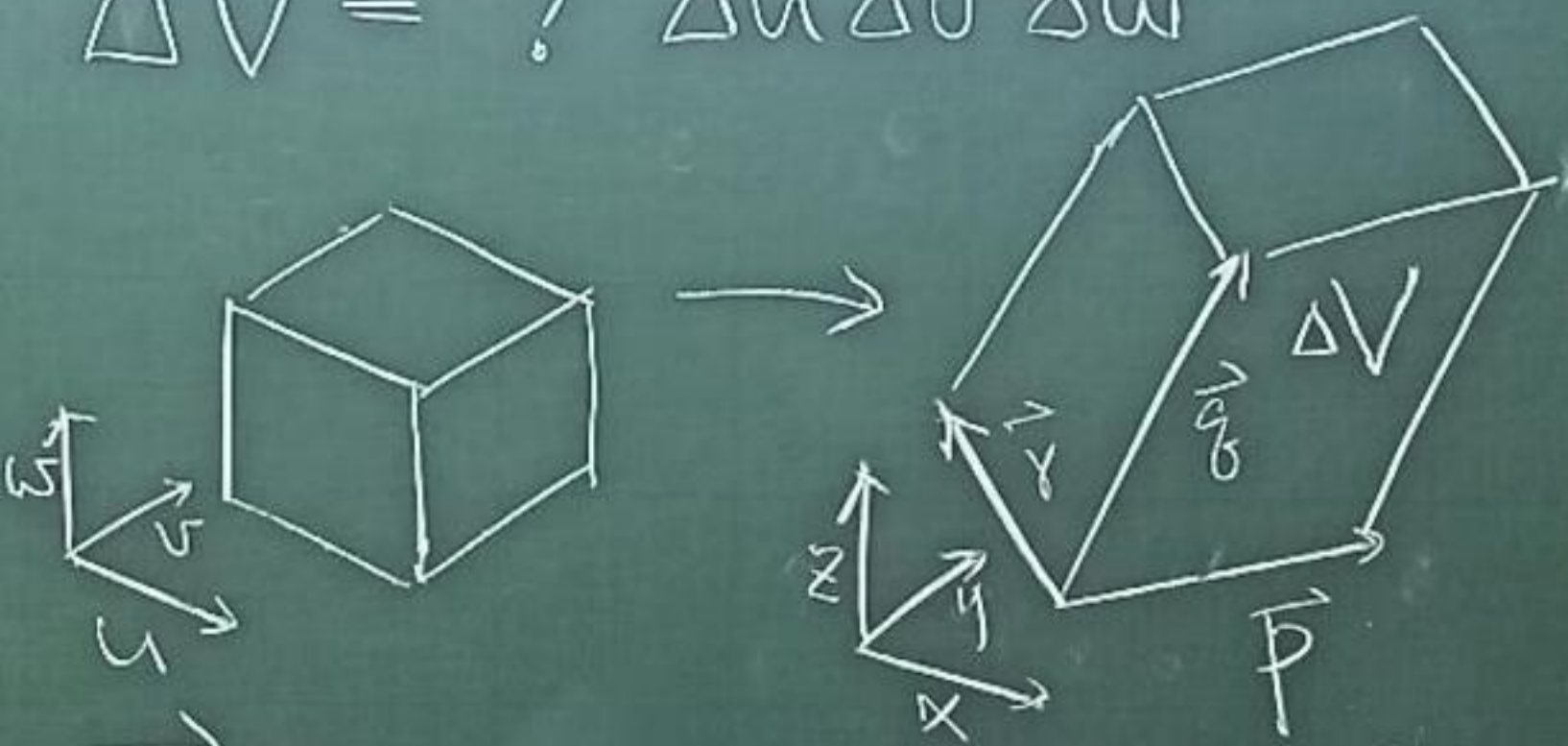


For Triple Integrals

$U(x, y, z), V(x, y, z), W(x, y, z)$

$$\Delta V = ? \quad \Delta U \Delta V \Delta W$$



$$\vec{p} = \mathbb{X}(u+\Delta u, v, w) - \mathbb{X}(u, v, w) \approx \mathbb{X}_u \Delta u$$

$$\vec{q} = \mathbb{X}(u, v+\Delta v, w) - \mathbb{X}(u, v, w) \approx \mathbb{X}_v \Delta v$$

$$\vec{r} = \mathbb{X}(u, v, w+\Delta w) - \mathbb{X}(u, v, w) \approx \mathbb{X}_w \Delta w$$

$$\Delta V = \left| \vec{p} \times \vec{q} \cdot \vec{r} \right| = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right| \Delta u \Delta v \Delta w$$
$$= |J| \Delta u \Delta v \Delta w$$

Summary

Step 1 Find suitable

$u(x, y, z), v(x, y, z), w(x, y, z)$
for the problem.

Step 2 Solve for $x(u, v, w)$
 $y(u, v, w), z(u, v, w)$

Step 3 $J = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$

$$\Delta V = |J| \Delta u \Delta v \Delta w$$

Eq. 2: Evaluate $\int_{z=0}^3 \int_{y=0}^4 \int_{x=\frac{y}{2}}^{\frac{y}{2}+1} \left(\frac{2x-y}{z} + \frac{z}{3} \right) dx dy dz$

Using $u = x - \frac{y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$

Sol

$x = u + v$, $y = 2v$, $z = 3w$

$dw = \int_{z=0}^3 = \int_{w=0}^1$, $dv = \int_{y=0}^4 = \int_{v=0}^2$

$du = \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}$
 $x = \frac{y}{2} \Leftrightarrow u = 0$
 $x = \frac{y}{2} + 1 \Leftrightarrow u = 1$

$J = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 6$

$$I = \int_{w=0}^1 \int_{v=0}^2 \int_{u=0}^1 (u+w) \underline{6 \, du \, dv \, dw}$$

$$= 6 \left(\iiint u \, du \, dv \, dw + \iiint w \, du \, dv \, dw \right)$$

$$= 6 \left(\int_{w=0}^1 \int_{v=0}^2 \left. \frac{u^2}{2} \right|_0^1 du \, dv \, dw + \int_{v=0}^2 \int_{u=0}^1 \left. \left(\frac{w^2}{2} \right) \right|_0^1 du \, dv \right)$$

$$= 6 \left(\frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 2 \right)$$

$$= 12$$

Eg 3: Spherical Coordinate

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$J = \det \begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{pmatrix} = \rho^2 \sin \phi \quad (\text{check!})$$