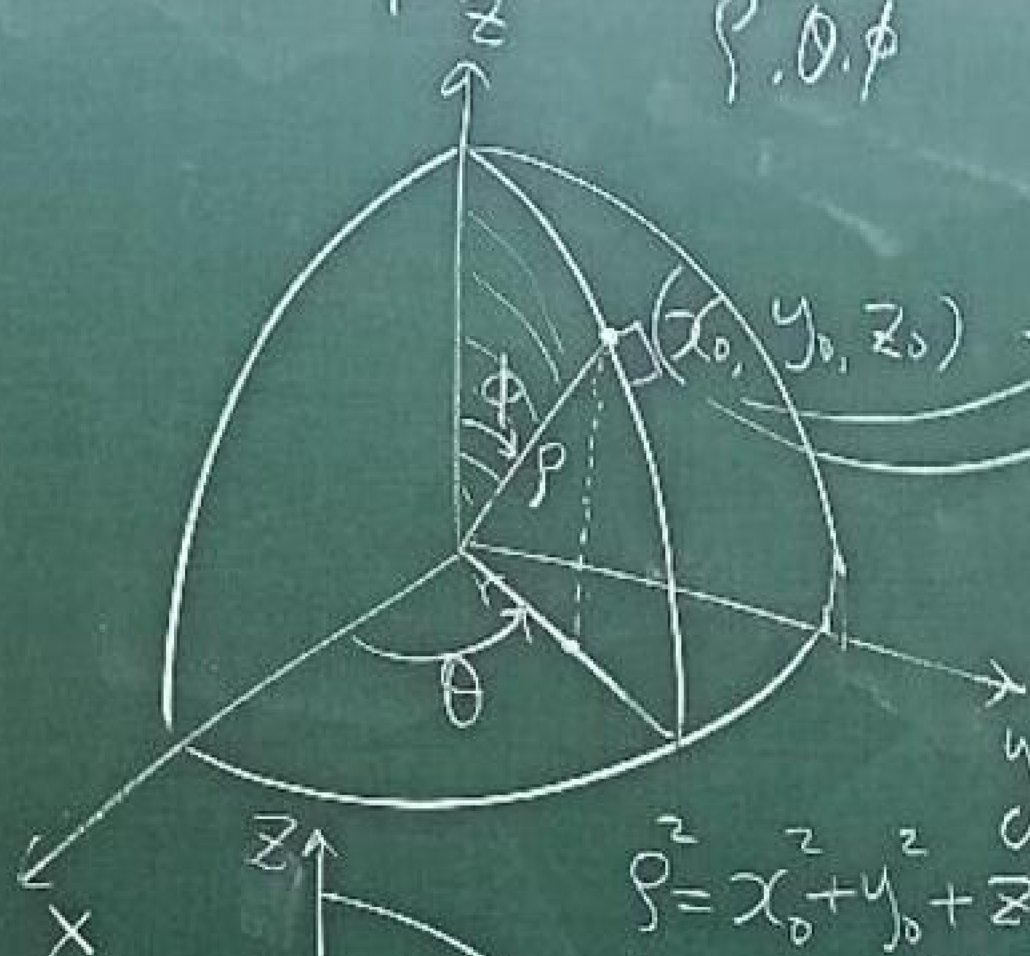
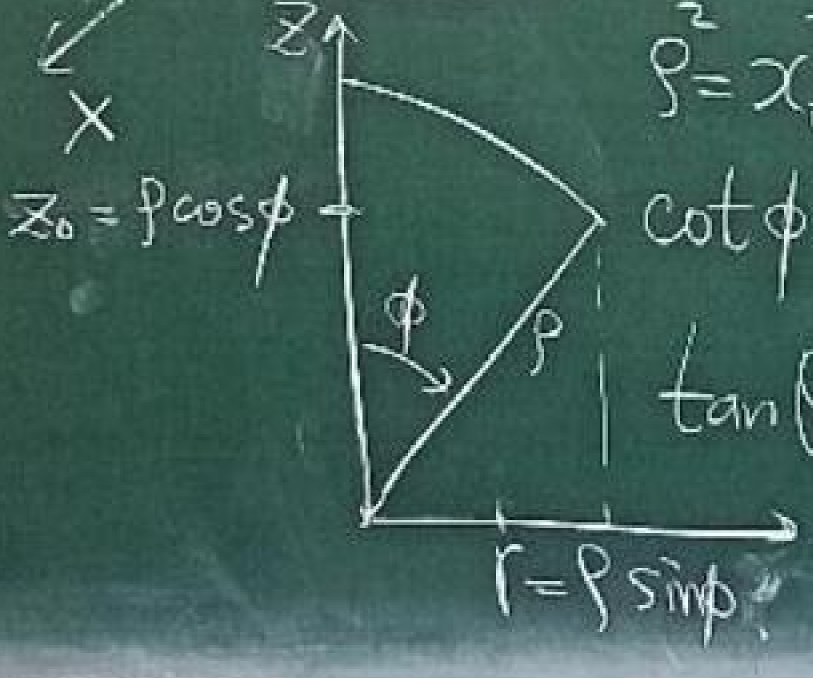
II Spherical Coordinates

ρ, θ, ϕ



$$\Delta A = \rho^2 \sin\phi \, \Delta\theta \, \Delta\phi$$

$$\Delta V = \Delta A \, \Delta\rho = \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$$



$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot\phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan\theta = \frac{y_0}{x_0}$$

$$x_0 = r\cos\theta = \rho\sin\phi\cos\theta$$

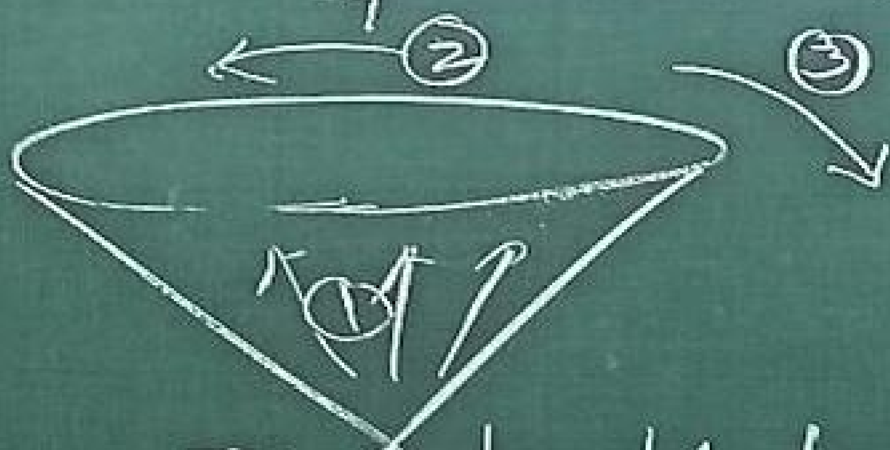
$$y_0 = r\sin\theta = \rho\sin\phi\sin\theta$$

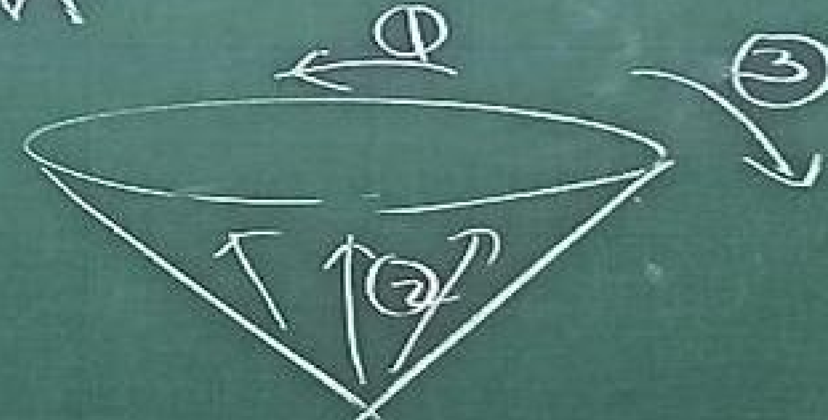
$$z_0 = \rho\cos\phi$$

Cross Sections for various order of integration

Case I: $d\rho\, d\theta\, d\phi$ or $d\theta\, d\rho\, d\phi$

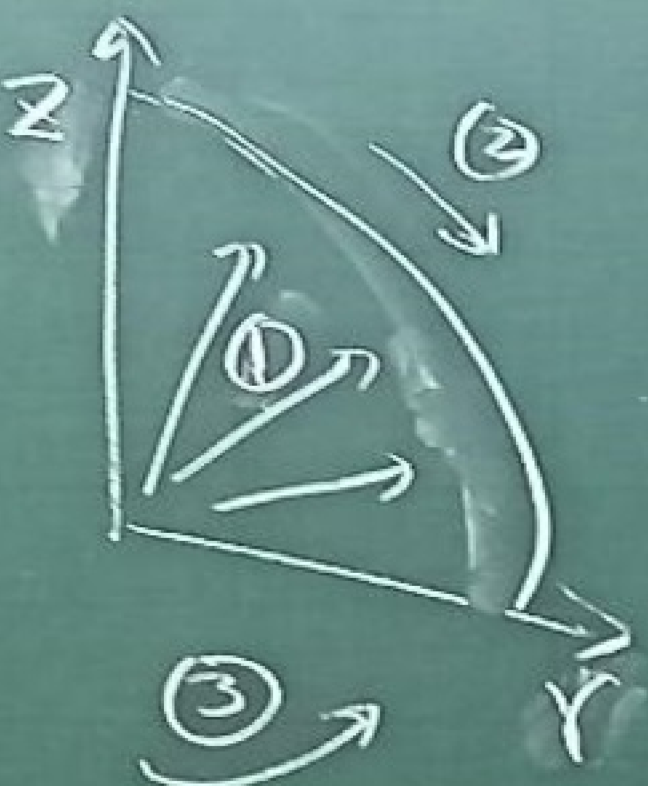
$\phi = $ constant





Case II $d\theta\, d\phi\, d\rho$ $\quad$ $d\phi\, d\theta\, d\rho$

Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$

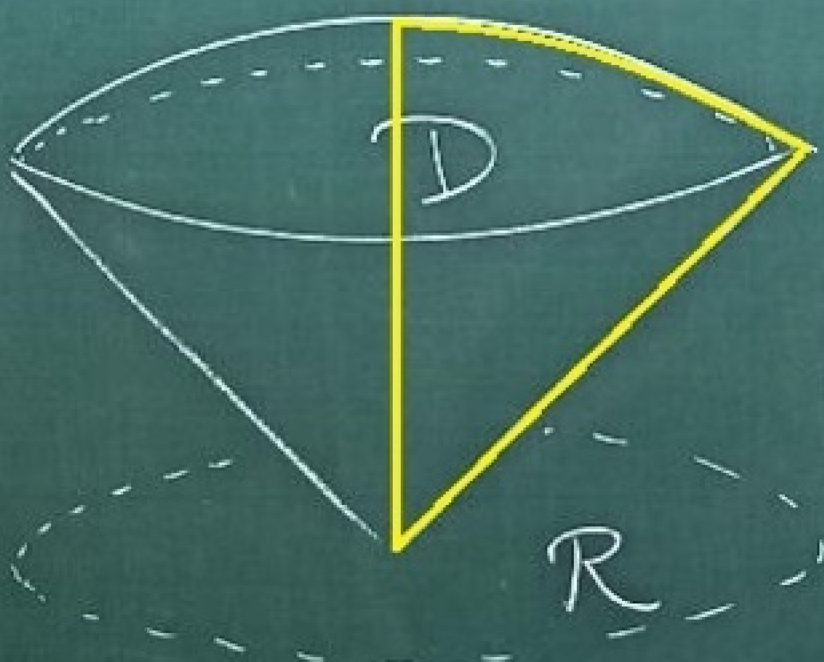


Easiest case in general

Must learn!

Example. $D = \left\{ \begin{array}{l} x^2+y^2+z^2 \le 1 \\ z \geq \sqrt{\dfrac{x^2+y^2}{3}} \end{array} \right\}$

Find volume of D



Sol (I). Cylindrical coordinate.

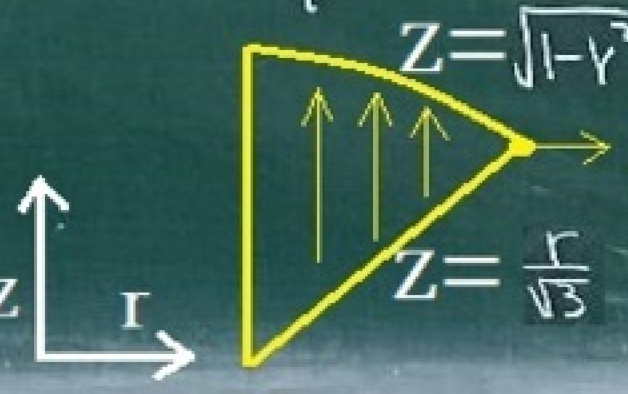
$$R = \left\{ (x,y) \;\; \exists z \in \mathbb{R} \;\; \sqrt{\frac{x^2+y^2}{3}} \le z \le \sqrt{1-(x^2+y^2)} \right\}$$

$$= \left\{ \frac{x^2+y^2}{3} \le 1-(x^2+y^2) \right\}$$

$$= \left\{ x^2+y^2 \le \frac{3}{4} \right\}$$

$$V = \iint_R \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\, dA$$

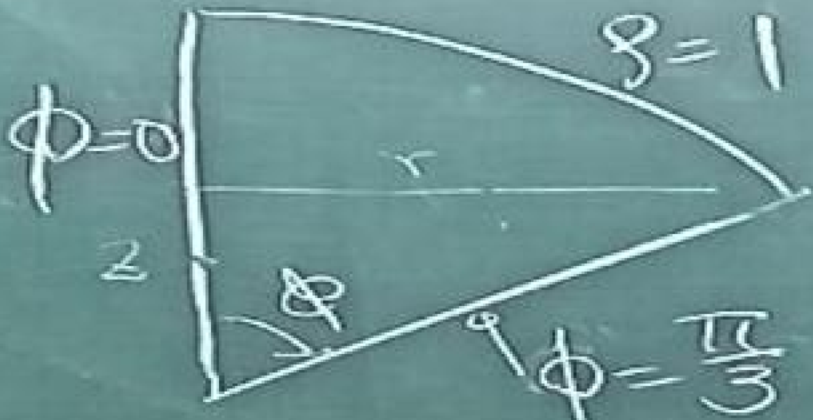
$$= \int_0^{2\pi} \int_{r=0}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{3}\right) r\, dr\, d\theta$$



$$\frac{x^2+y^2}{3} = 1 - x^2 - y^2$$

II Spherical Coordinate

$(x^2+y^2+z^2=1)$



$\left(\frac{r}{z}=\sqrt{3},\ \tan\phi=\sqrt{3}\right)$

$z=\sqrt{\frac{x^2+y^2}{3}}$

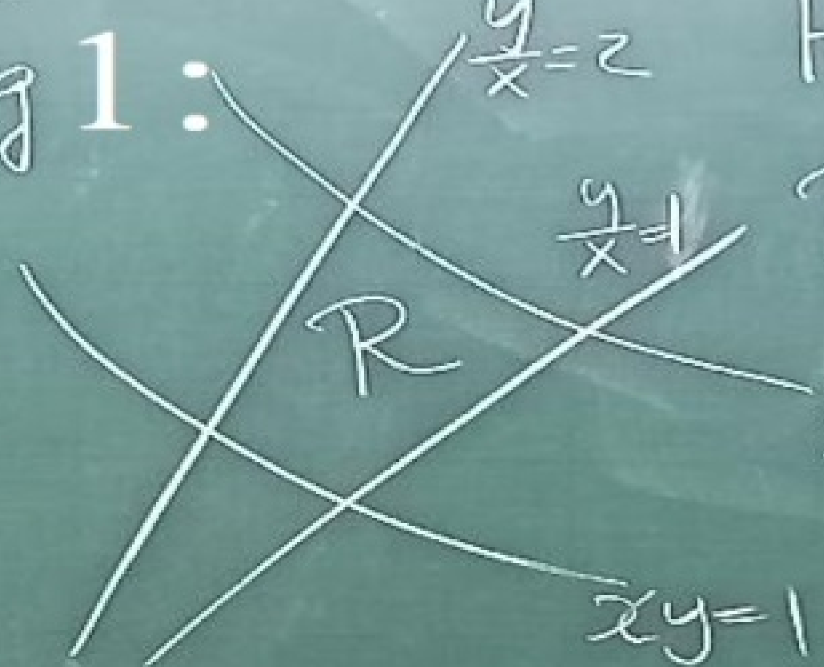
$$D=\left\{\begin{array}{l} 0\le\rho\le 1 \\ 0\le\phi\le\frac{\pi}{3} \\ 0\le\theta\le 2\pi \end{array}\right\}$$

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\underbrace{\frac{1}{3}}_{d\rho}\underbrace{\left(1-\frac{1}{2}\right)}_{d\phi}\underbrace{2\pi}_{d\theta}=\frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg 1:



Find area of

$$R=\left\{\begin{array}{l}1\le xy\le 2\\ 1\le \frac{y}{x}\le 2\end{array}\right\}=\left\{\begin{array}{l}1\le u\le 2\\ 1\le v\le 2\end{array}\right\}$$

$$A=\int_{v=1}^{2}\int_{u=1}^{2} dA$$

$\underline{\underline{Q}}$: $dA=\underline{?}\,du\,dv$

$\underline{Ans}$: $dA=\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right|du\,dv$

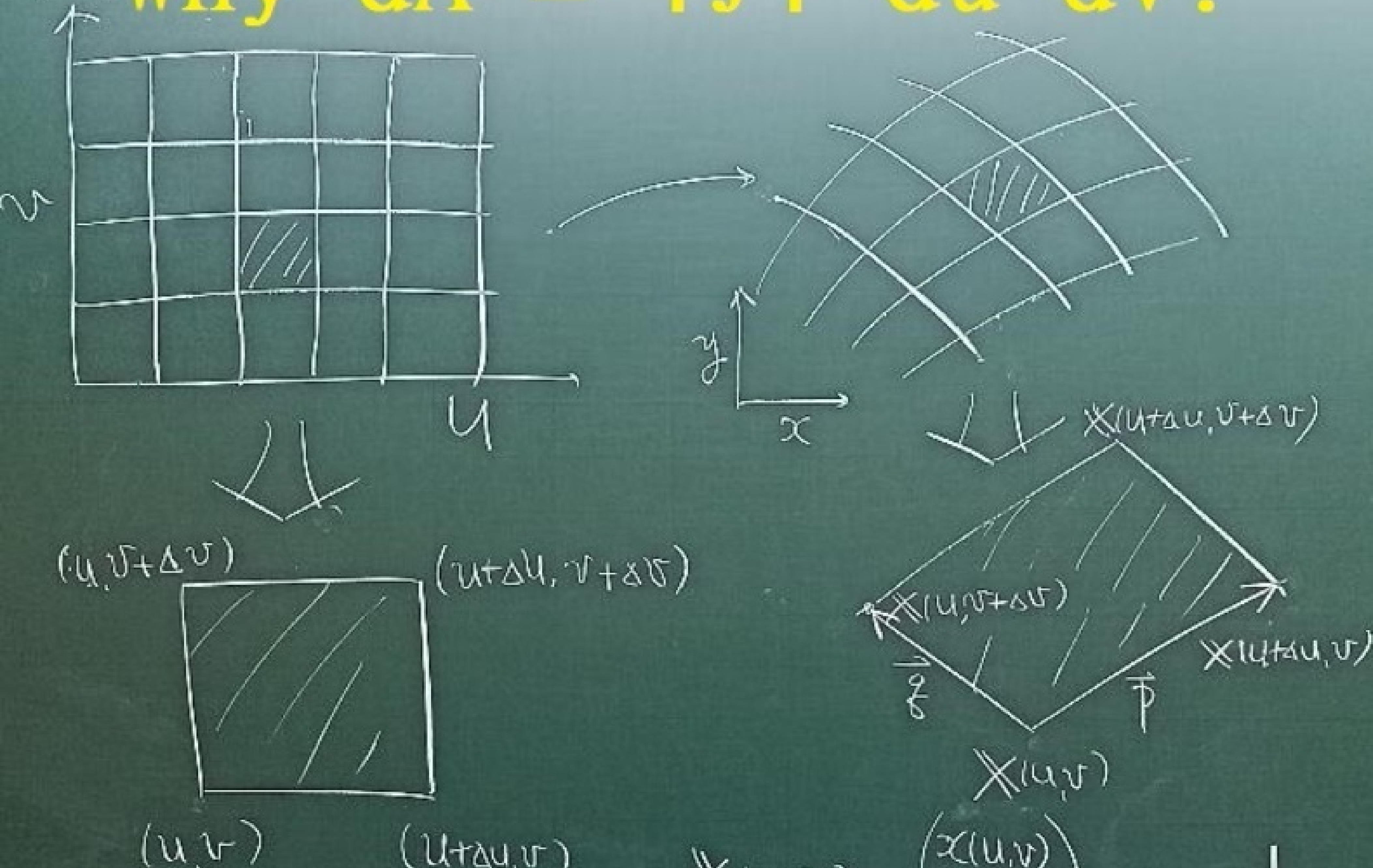
$$\underset{J}{\overset{\parallel}{\frac{\partial(x,y)}{\partial(u,v)}}}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$

Need $\left\{\begin{array}{l}u=xy\\ v=\frac{y}{x}\end{array}\right.$

$\rightarrow$ $x=x(u,v)=\sqrt{\frac{u}{v}}$

$y=y(u,v)=\sqrt{uv}$

Why dA = |J| du dv:



$$\mathbb{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}, \quad \cdots \text{ etc.}$$

$$\mathbb{X}(u+\Delta u, v) = \begin{pmatrix} x(u+\Delta u, v) \\ y(u+\Delta u, v) \end{pmatrix}$$

$$\Delta A \cong |\vec{p} \times \vec{q}|$$

$$= \left| \left(\mathbb{X}(u+\Delta u) - \mathbb{X}(u) \right) \times \left(\mathbb{X}(u, v+\Delta v) - \mathbb{X}(u,v) \right) \right|$$

$$\underline{\underline{\text{M.V.T.}}} \left| \Delta u \begin{pmatrix} \frac{\partial x}{\partial u}(u+c_{11}\Delta u, v) \\ \frac{\partial y}{\partial u}(u+c_{21}\Delta u, v) \end{pmatrix} \times \Delta v \begin{pmatrix} \frac{\partial x}{\partial v}(u, v+c_{12}\Delta v) \\ \frac{\partial y}{\partial v}(u, v+c_{22}\Delta v) \end{pmatrix} \right|$$

$$= \Delta u \Delta v \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \left(0, 0, \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

we sometimes write

$$(a, b) \times (c, d) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \Delta A = |J| \Delta u \Delta v$$

$$J = \det\left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\right)$$

$$= \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eg 1:

$$u = xy \qquad v = \frac{y}{x}$$

$$A = \int_{u=1}^{2}\int_{v=1}^{2} dA$$

$$x = \sqrt{\frac{u}{v}} \qquad y = \sqrt{uv}$$

$$= \int_1^2\int_1^2 |J| \, du\, dv$$

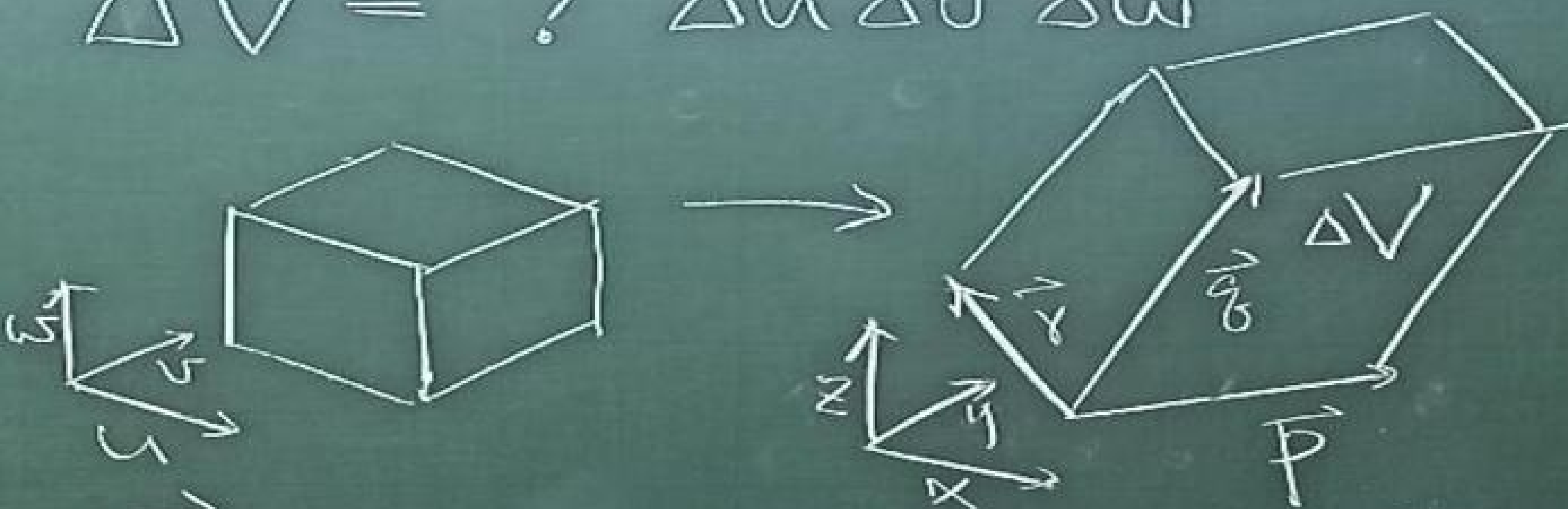
$$= \int_1^2\int_1^2 \left|\det\begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix}\right| du\, dv$$

$$= \int_1^2\int_1^2 \left|\frac{1}{2v}\right| du\, dv = \ln 2/2$$

For Triple Integrals

$u(x,y,z),\ v(x,y,z),\ w(x,y,z)$

$\Delta V = \ ?\ \Delta u \Delta v \Delta w$



$$\vec{p} = \mathbb{X}(u+\Delta u, v, w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_u \Delta u$$

$$\vec{q} = \mathbb{X}(u, v+\Delta v, w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_v \Delta v$$

$$\vec{r} = \mathbb{X}(u, v, w+\Delta w) - \mathbb{X}(u,v,w) \cong \mathbb{X}_w \Delta w$$

$$\Delta V = \left| \vec{p} \times \vec{q} \cdot \vec{r} \right| = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right| \Delta u \Delta v \Delta w$$

$$= |J| \Delta u \Delta v \Delta w$$

Summary

Step 1 Find suitable $u(x,y,z), v(x,y,z), w(x,y,z)$ for the problem.

Step 2 Solve for $x(u,v,w)$, $y(u,v,w)$, $z(u,v,w)$

Step 3. $J = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$

$$\Delta V = |J| \Delta u \Delta v \Delta w$$

Eg 2:

Evaluate $$\int_{z=0}^{3}\int_{y=0}^{4}\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

using $u=x-\frac{y}{2}$, $v=\frac{y}{2}$, $w=\frac{z}{3}$

Sol

$x=u+v$, $y=2v$, $z=3w$

dw: $\int_{z=0}^{3}=\int_{w=0}^{1}$, dv: $\int_{y=0}^{4}=\int_{v=0}^{2}$

du: $\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}$ $\quad x=\frac{y}{2}\Leftrightarrow u=0$

$x=\frac{y}{2}+1\Leftrightarrow u=1$

$$J=\det\begin{pmatrix}x_u & x_v & x_w\\ y_u & y_v & y_w\\ z_u & z_v & z_w\end{pmatrix}=\det\begin{pmatrix}1&1&0\\0&2&0\\0&0&3\end{pmatrix}$$

$=6$

$$I = \int_{w=0}^{1}\int_{v=0}^{2}\int_{u=0}^{1} (u+w)\underline{6\,du\,dv\,dw}$$

$$= 6\left(\iiint u\,du\,dv\,dw + \iiint w\,du\,dv\,dw\right)$$

$$= 6\left(\int_{w=0}^{1}\int_{v=0}^{2} \frac{u^2}{2}\Big|_0^1 dv\,dw + \int_{v=0}^{2}\int_{u=0}^{1} \left(\frac{w^2}{2}\right)\Big|_0^1 du\,dv\right)$$

$$= 6\left(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 2\right)$$

$$= 12$$

Eg3: Spherical Coordinate

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$J = \det\begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix}$$

$$= \det\begin{pmatrix} S_\phi C_\theta & \rho C_\phi C_\theta & -\rho S_\phi S_\theta \\ S_\phi S_\theta & \rho C_\phi S_\theta & \rho S_\phi C_\theta \\ C_\phi & -\rho S_\phi & 0 \end{pmatrix} = \rho^2\sin\phi$$

(check!)