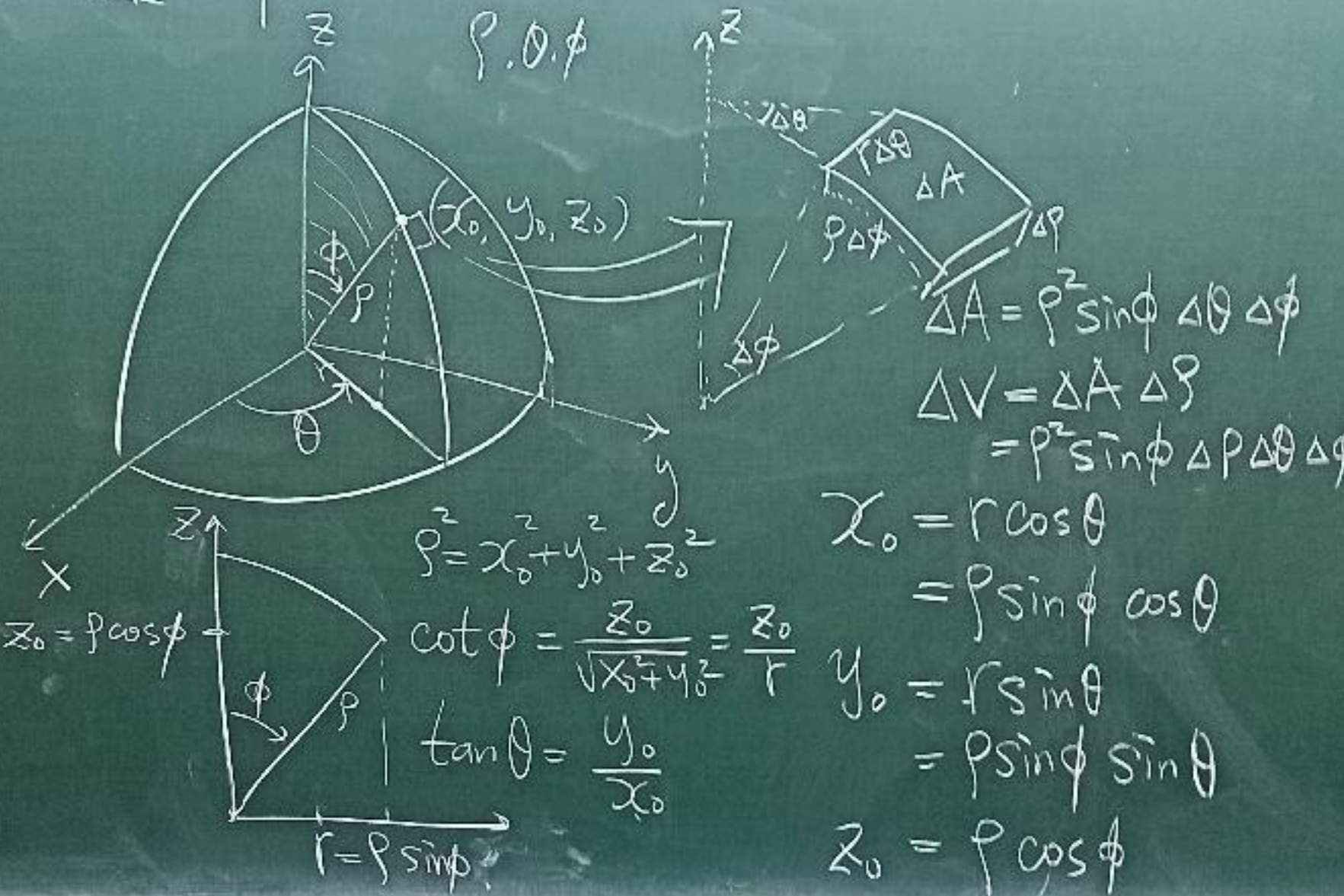


II Spherical Coordinates



P.O.P

$$\Delta A = \rho^2 \sin \phi \Delta \theta \Delta \phi$$

$$\Delta V = \Delta A \Delta \rho$$

$$= \rho^2 \sin \phi \Delta \rho \Delta \theta \Delta \phi$$

$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot \phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan \theta = \frac{y_0}{x_0}$$

$$r = \rho \sin \phi$$

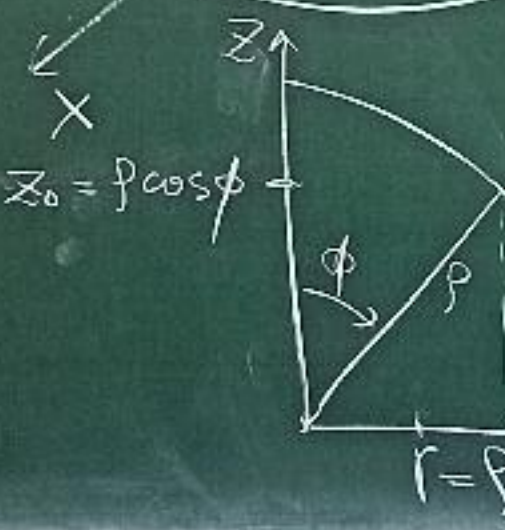
$$x_0 = r \cos \theta$$

$$= \rho \sin \phi \cos \theta$$

$$y_0 = r \sin \theta$$

$$= \rho \sin \phi \sin \theta$$

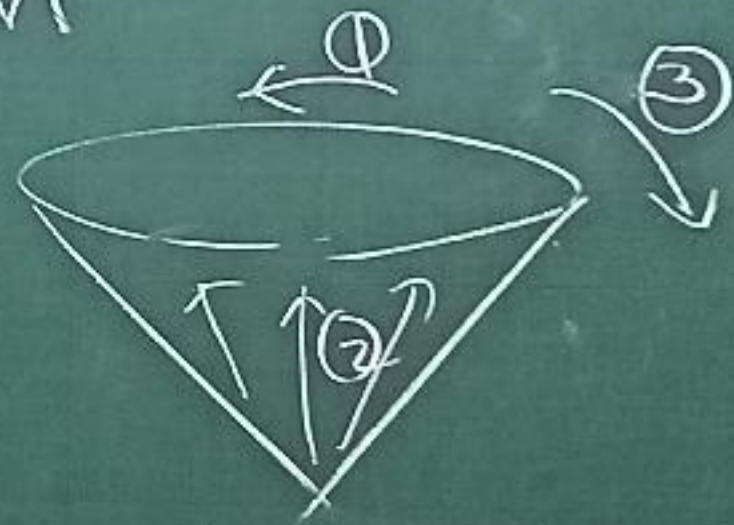
$$z_0 = \rho \cos \phi$$



Cross Sections for various order of integration

Case I: $dr d\theta d\phi$ or $d\theta dr d\phi$

$\phi = \text{constant}$



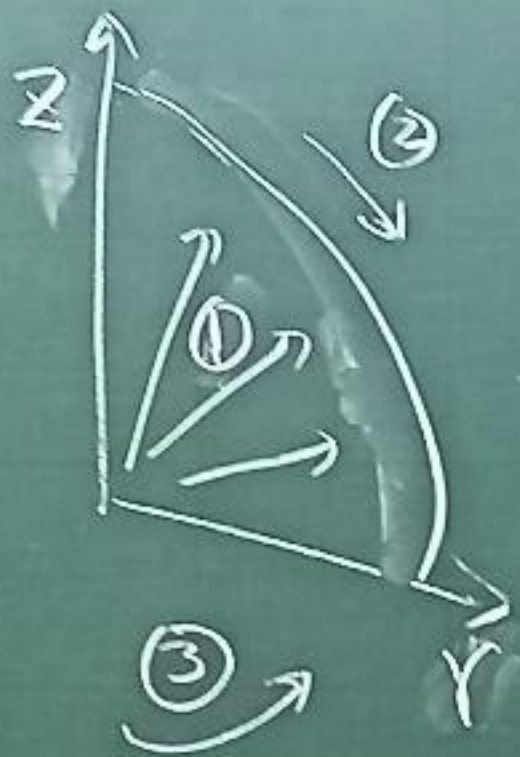
Case II $d\theta d\phi dr$



$d\phi d\theta dr$

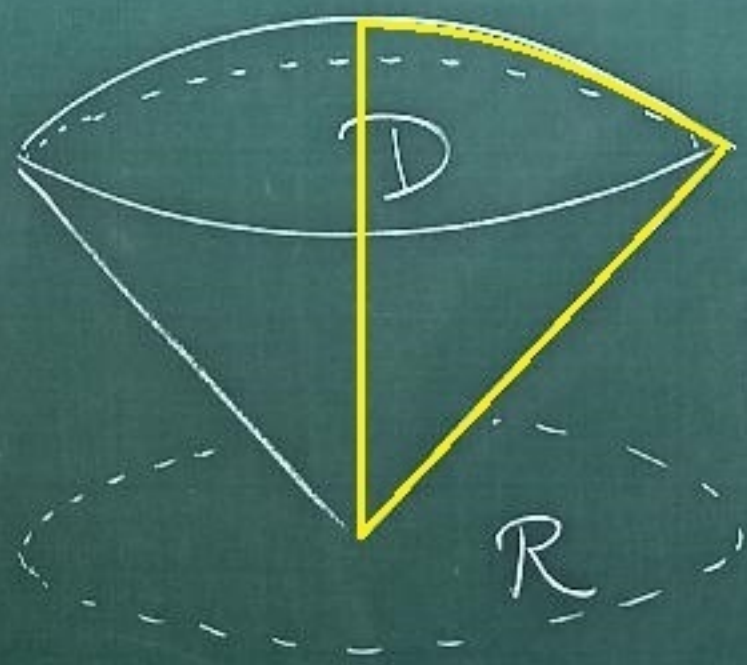


Case III $d\rho d\phi d\theta$, $d\phi d\rho d\theta$



Easiest case in general
Must learn!

Example. $D = \left\{ \begin{array}{l} x^2 + y^2 + z^2 \leq 1 \\ z \geq \sqrt{\frac{x^2 + y^2}{3}} \end{array} \right\}$
 Find volume of D



Sol (I). Cylindrical coordinate.

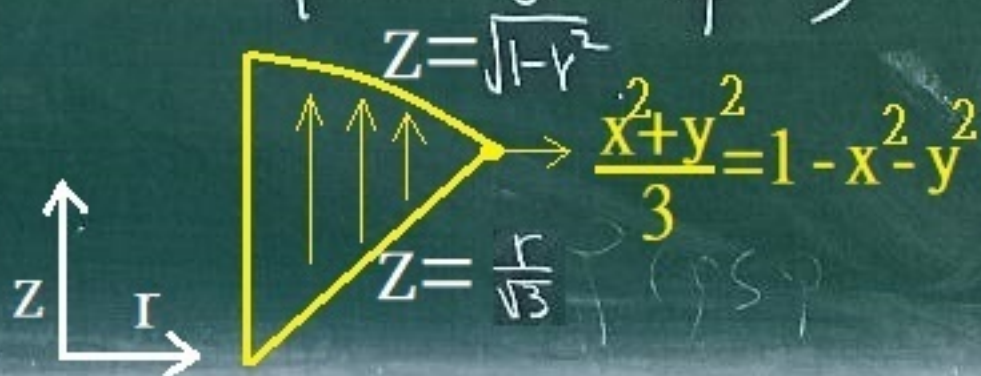
$$R = \left\{ \begin{array}{l} (x, y) \exists z \in \mathbb{R} \\ \sqrt{\frac{x^2 + y^2}{3}} \leq z \leq \sqrt{1 - (x^2 + y^2)} \end{array} \right\}$$

$$= \left\{ \frac{x^2 + y^2}{3} \leq 1 - (x^2 + y^2) \right\}$$

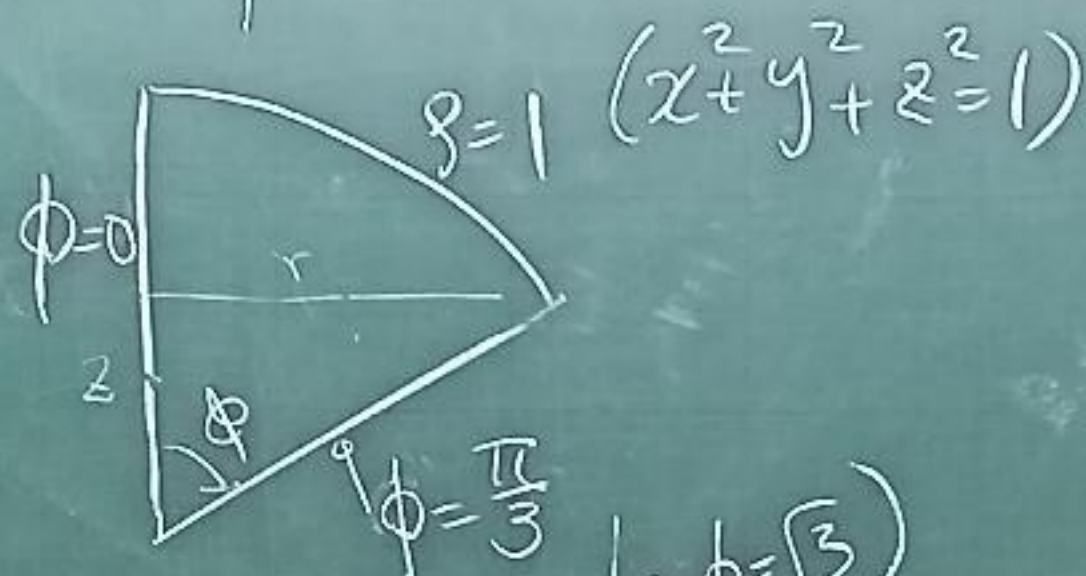
$$= \left\{ x^2 + y^2 \leq \frac{3}{4} \right\}$$

$$V = \iint_R \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz dA$$

$$= \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}} \right) r dr d\theta$$



II Spherical Coordinate



$$\left(\frac{x}{z} = \sqrt{3}, \tan \phi = \sqrt{3} \right)$$
$$z = \sqrt{\frac{x^2 + y^2}{3}}$$


$$D = \left\{ \begin{array}{l} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \frac{\pi}{3} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left(\frac{1}{3} - \frac{1}{2} \right) d\phi \, d\theta = \frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg 1: $\frac{y}{x}=z$ Find area of $R = \left\{ \begin{array}{l} 1 \leq xy \leq 2 \\ 1 \leq \frac{y}{x} \leq 2 \end{array} \right\}$



$xy=2 = \left\{ \begin{array}{l} 1 \leq u \leq 2 \\ 1 \leq v \leq 2 \end{array} \right\}$

$xy=1$

$$A = \int_{v=1}^2 \int_{u=1}^2 dA, \quad \underline{Q}: dA = ? du dv$$

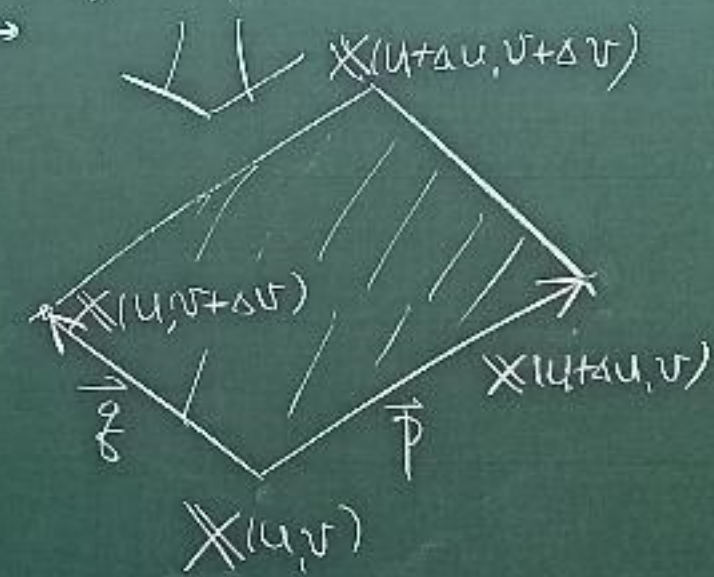
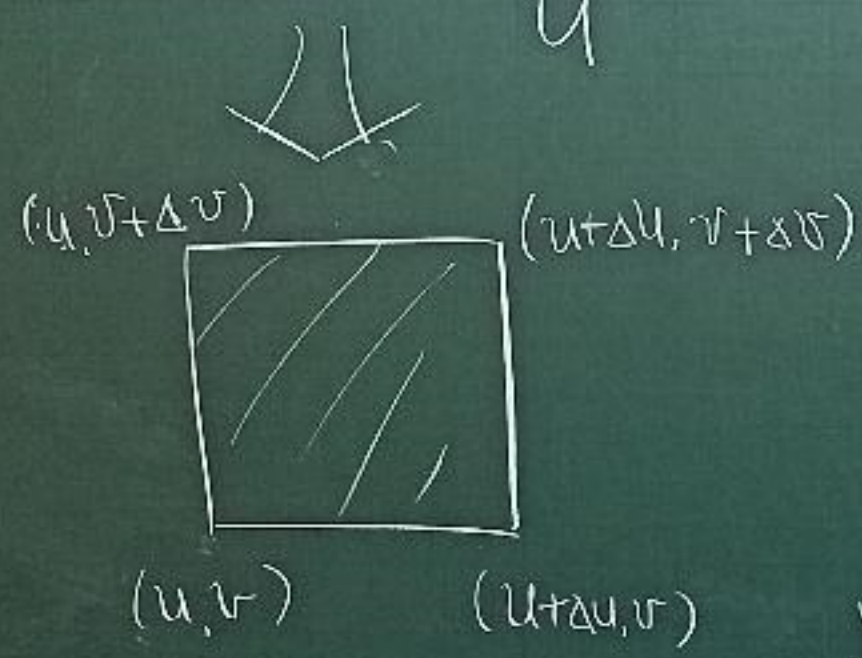
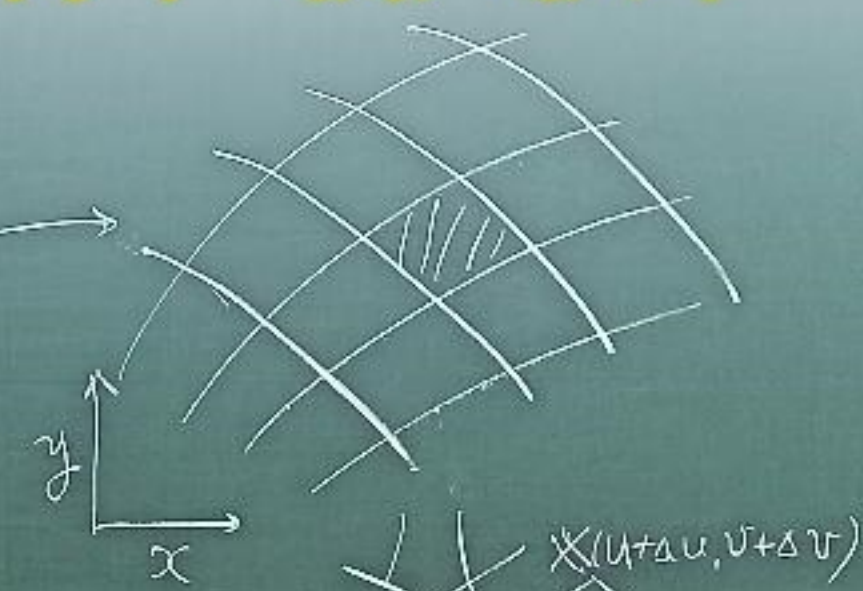
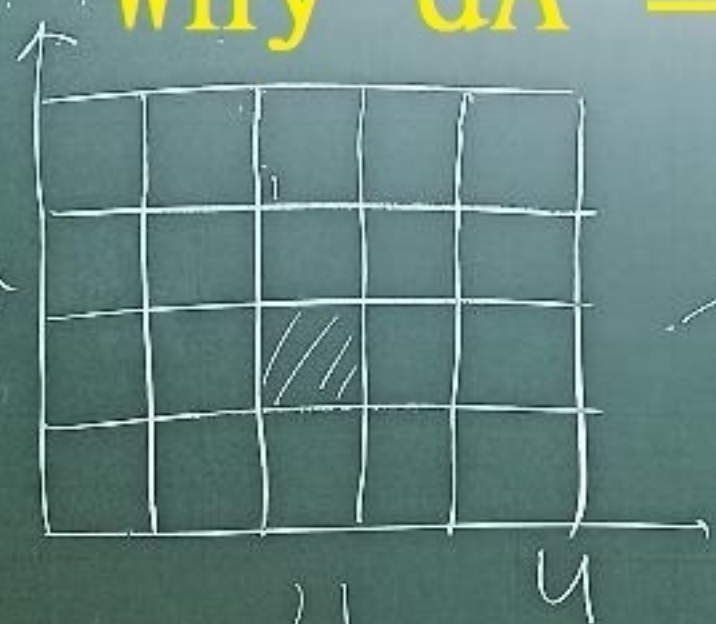
Ans: $dA = \left| \det \left(\frac{\partial(x,y)}{\partial(u,v)} \right) \right| du dv$

$$\frac{\partial(x,y)}{\partial(u,v)} \stackrel{J}{=} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Need $\begin{cases} u=xy \\ v=\frac{y}{x} \end{cases}$

$$\rightarrow \begin{cases} x = x(u,v) = \sqrt{\frac{u}{v}} \\ y = y(u,v) = \sqrt{uv} \end{cases}$$

Why $dA = |J| du dv$:



$$X(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ \dots \end{pmatrix} \text{ etc.}$$

$$X(u + \Delta u, v) = \begin{pmatrix} x(u + \Delta u, v) \\ y(u + \Delta u, v) \\ \dots \end{pmatrix}$$

$$\Delta A \approx |\vec{P} \times \vec{q}|$$

$$= \left| \left(X(u+\Delta u) - X(u) \right) \times \left(X(u, v+\Delta v) - X(u, v) \right) \right|$$

$$\underline{\text{M.V.T.}} \left| \Delta u \begin{pmatrix} \frac{\partial x}{\partial u}(u + C_{11}\Delta u, v) \\ \frac{\partial y}{\partial u}(u + C_{21}\Delta u, v) \end{pmatrix} \times \Delta v \begin{pmatrix} \frac{\partial x}{\partial v}(u, v + C_{12}\Delta v) \\ \frac{\partial y}{\partial v}(u, v + C_{22}\Delta v) \end{pmatrix} \right|$$

$$= \Delta u \Delta v \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = (0, 0, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

We sometimes write

$$(a, b) \times (c, d) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \Delta A = |J| \Delta u \Delta v$$

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eq 1:

$$A = \int_{u=1}^2 \int_{v=1}^2 dA \quad \begin{array}{l} u = xy \\ v = \frac{y}{x} \end{array}$$

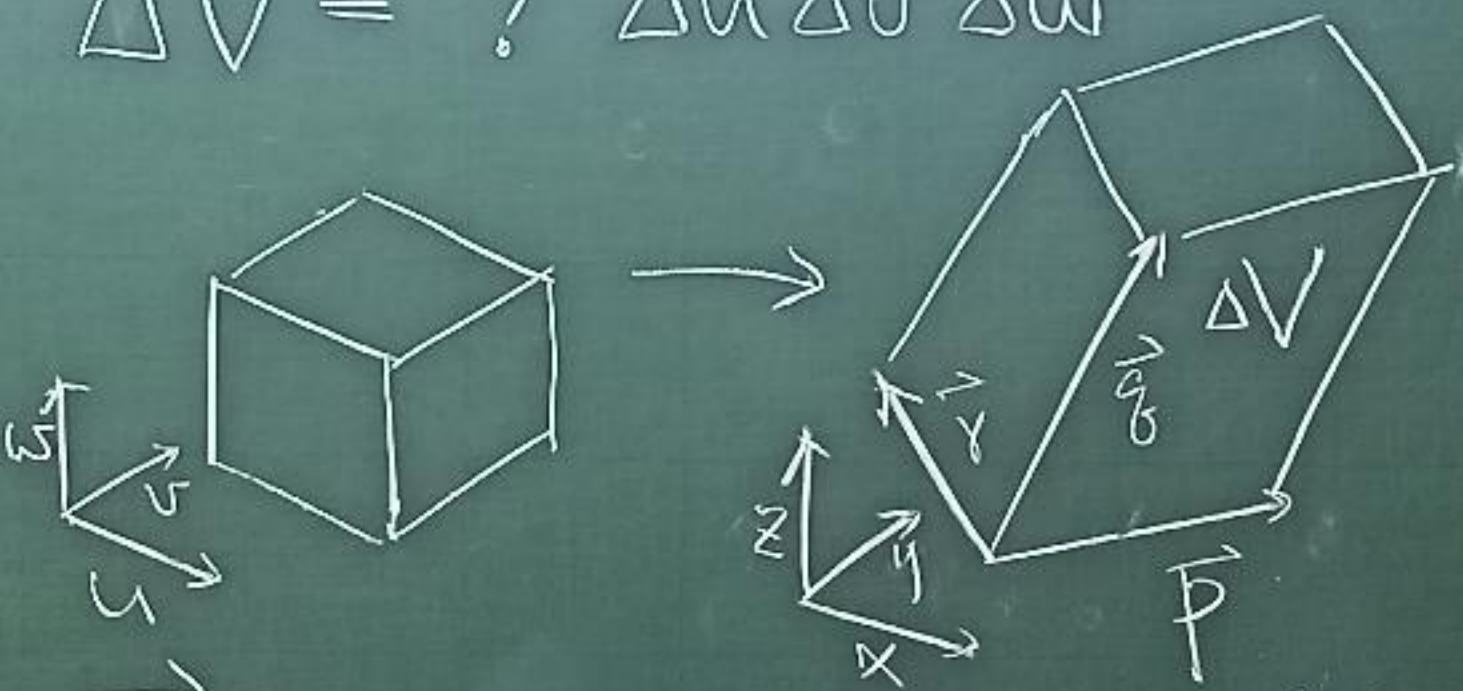
$$= \int_1^2 \int_1^2 |J| du dv \quad \begin{array}{l} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{array}$$

$$= \int_1^2 \int_1^2 \left| \det \begin{pmatrix} \frac{1}{2} \frac{1}{\sqrt{uv}} & \frac{1}{2} \sqrt{\frac{u}{v^3}} \\ \frac{1}{2} \sqrt{\frac{v}{u}} & \frac{1}{2} \sqrt{\frac{u}{v}} \end{pmatrix} \right| du dv$$
$$= \int_1^2 \int_1^2 \frac{1}{2v} du dv = \ln 2 / 2$$

For Triple Integrals

$U(x, y, z), V(x, y, z), W(x, y, z)$

$$\Delta V = ? \quad \Delta U \Delta V \Delta W$$



$$\vec{p} = X(u+\Delta u, v, w) - X(u, v, w) \approx X_u \Delta u$$

$$\vec{q} = X(u, v+\Delta v, w) - X(u, v, w) \approx X_v \Delta v$$

$$\vec{r} = X(u, v, w+\Delta w) - X(u, v, w) \approx X_w \Delta w$$

$$\Delta V = \left| \vec{p} \times \vec{q} \cdot \vec{r} \right| = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right| \Delta u \Delta v \Delta w$$
$$= |J| \Delta u \Delta v \Delta w$$

Summary

Step 1 Find suitable

$u(x, y, z), v(x, y, z), w(x, y, z)$
for the problem.

Step 2 Solve for $x(u, v, w)$
 $y(u, v, w), z(u, v, w)$

Step 3. $J = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$

$$\Delta V = |J| \Delta u \Delta v \Delta w$$

Eq. 2: Evaluate $\int_{z=0}^3 \int_{y=0}^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \left(\frac{2x-y}{z} + \frac{z}{3} \right) dx dy dz$

Using $u = x - \frac{y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$

Sol $x = u + v$, $y = 2v$, $z = 3w$

$dw = \int_{z=0}^3 = \int_{w=0}^1$, $dv = \int_{y=0}^4 = \int_{v=0}^2$

$du = \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}$

$x = \frac{y}{2} \Leftrightarrow u = 0$

$x = \frac{y}{2} + 1 \Leftrightarrow u = 1$

$J = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 6$

$$I = \int_{w=0}^1 \int_{v=0}^2 \int_{u=0}^1 (u+w) \underline{6 \, du \, dv \, dw}$$

$$= 6 \left(\iiint u \, du \, dv \, dw + \iiint w \, du \, dv \, dw \right)$$

$$= 6 \left(\int_{w=0}^1 \int_{v=0}^2 \left. \frac{u^2}{2} \right|_0^1 du \, dv \, dw + \int_{v=0}^2 \int_{u=0}^1 \left. \left(\frac{w^2}{2} \right) \right|_0^1 du \, dv \, dw \right)$$

$$= 6 \left(\frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 2 \right)$$

$$= 12$$

Eg 3: Spherical Coordinate

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$J = \det \begin{pmatrix} x_\theta & x_\phi & x_\rho \\ y_\theta & y_\phi & y_\rho \\ z_\theta & z_\phi & z_\rho \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{pmatrix} = \rho^2 \sin \phi \quad (\text{check!})$$