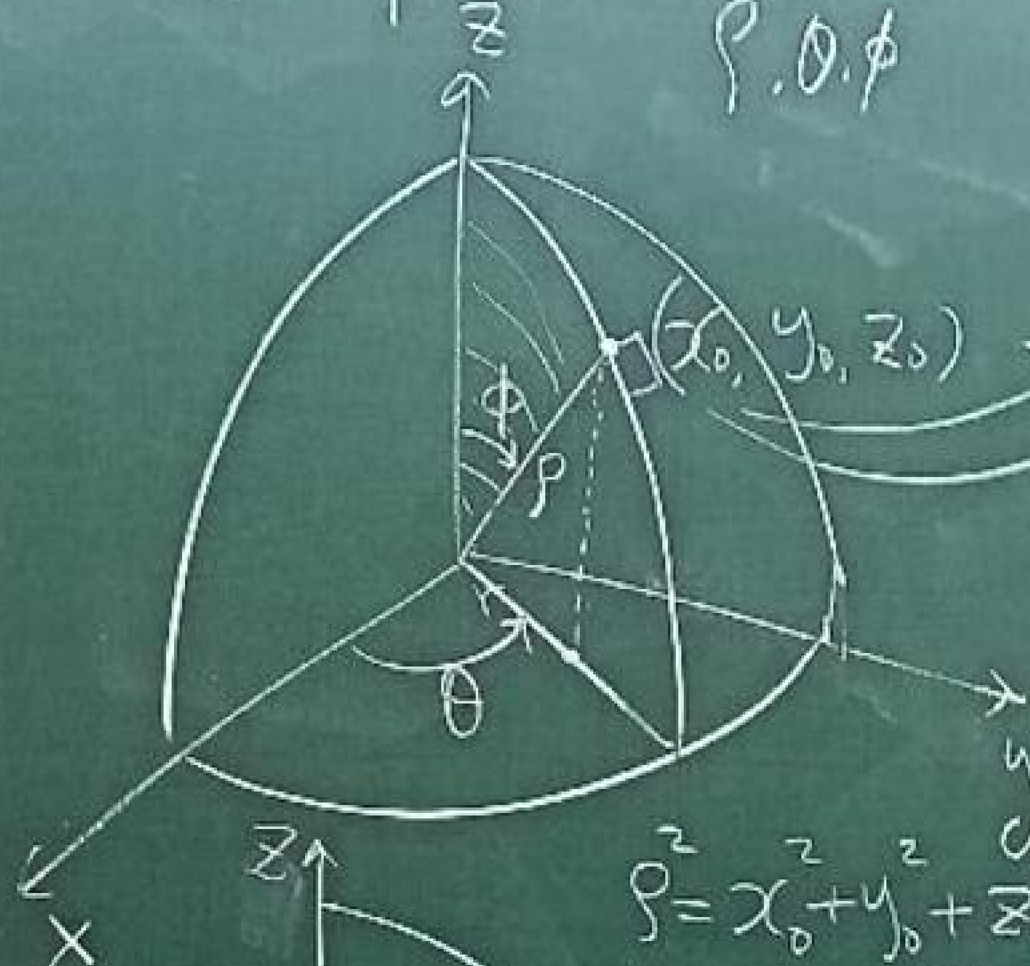
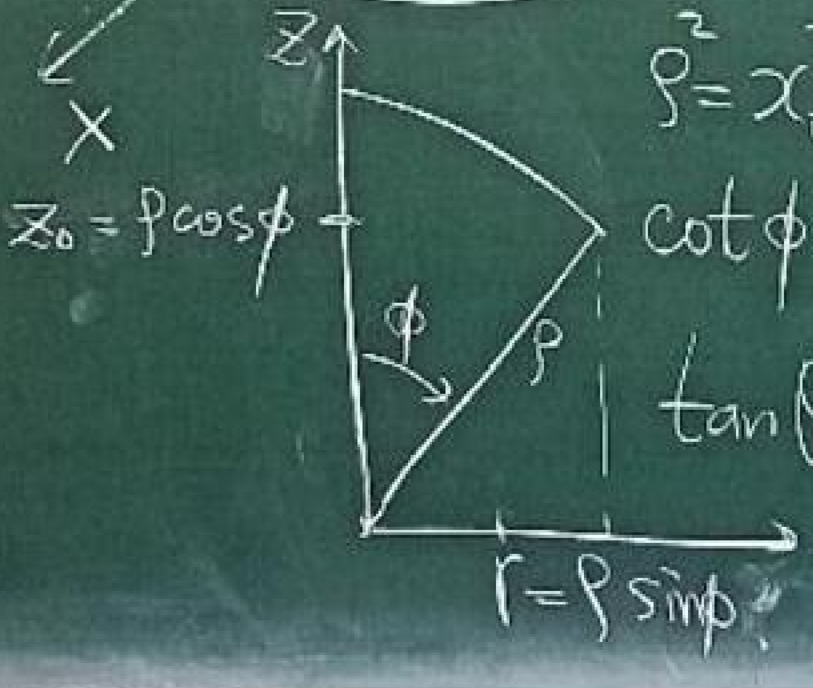
II Spherical Coordinates

ρ, θ, ϕ



$$\Delta A = \rho^2 \sin\phi \, \Delta\theta \, \Delta\phi$$

$$\Delta V = \Delta A \, \Delta\rho = \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$$



$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot\phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan\theta = \frac{y_0}{x_0}$$

$$x_0 = r\cos\theta = \rho\sin\phi\cos\theta$$

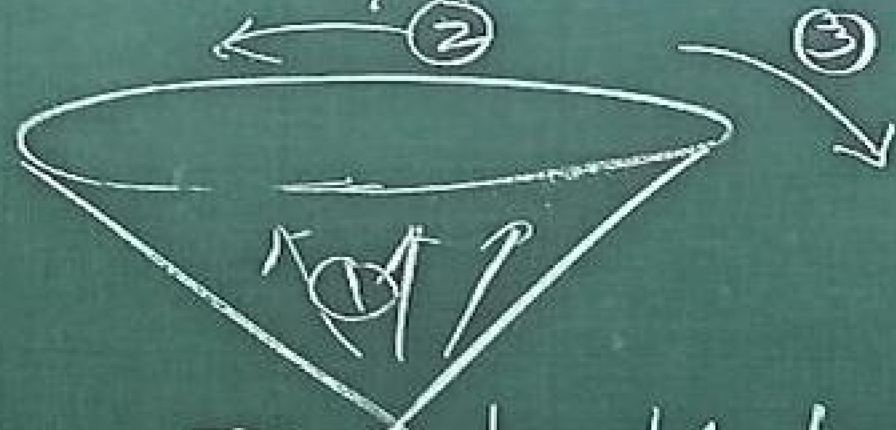
$$y_0 = r\sin\theta = \rho\sin\phi\sin\theta$$

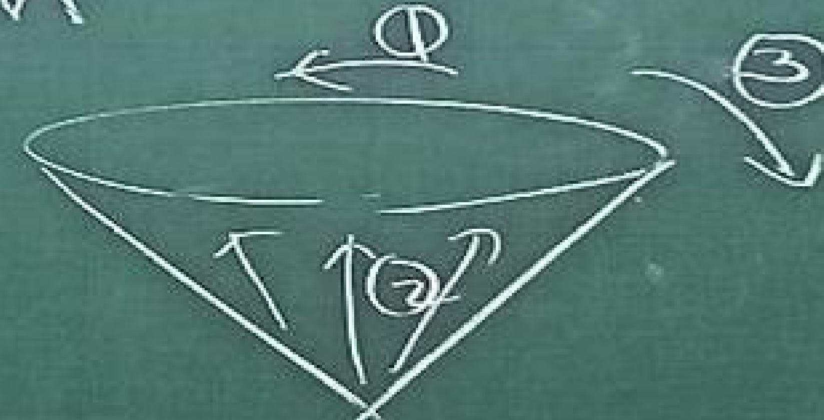
$$z_0 = \rho\cos\phi$$

Cross Sections for various order of integration

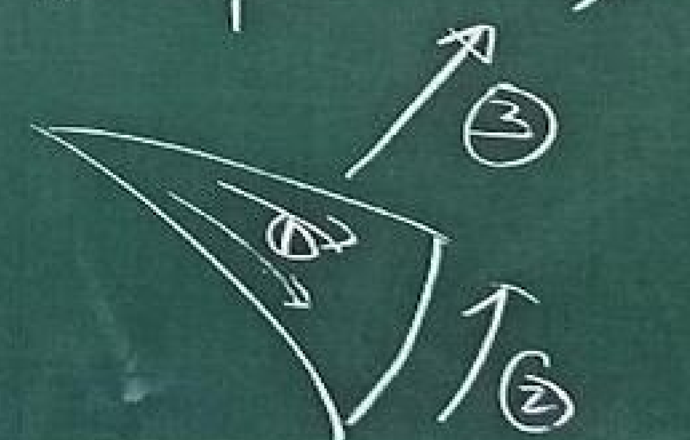
Case I: $d\rho\, d\theta\, d\phi$ or $d\theta\, d\rho\, d\phi$

$\phi = \text{constant}$

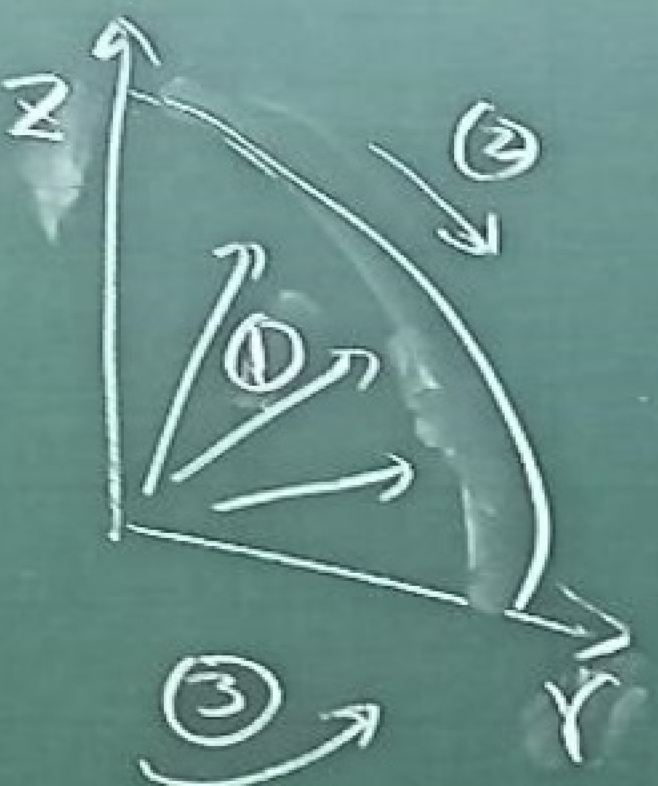




Case II $d\theta\, d\phi\, d\rho$ $\quad$ $d\phi\, d\theta\, d\rho$



Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$

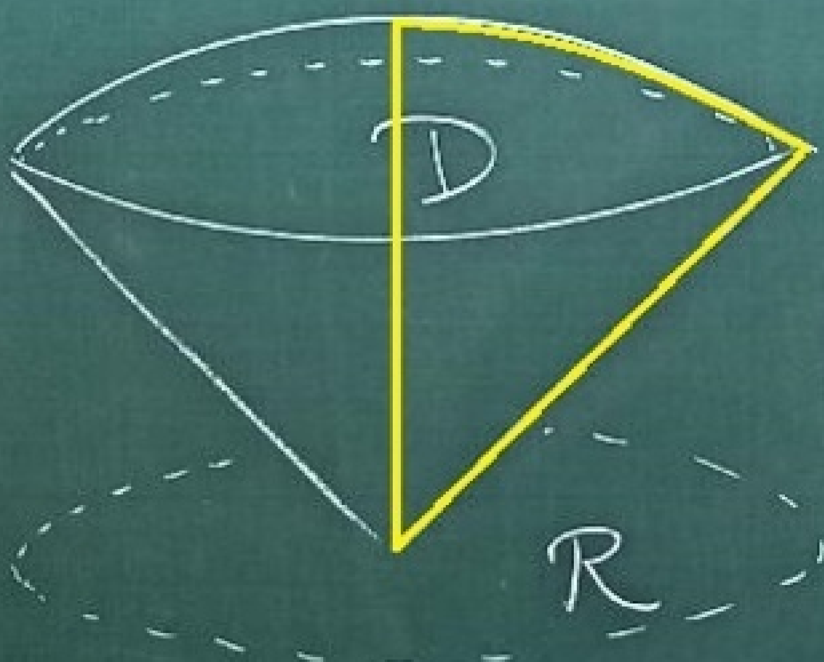


Easiest case in general

Must learn!

Example: $D = \left\{ \begin{array}{l} x^2+y^2+z^2 \le 1 \\ z \geq \sqrt{\dfrac{x^2+y^2}{3}} \end{array} \right\}$

Find volume of D



Sol (I): Cylindrical coordinate.

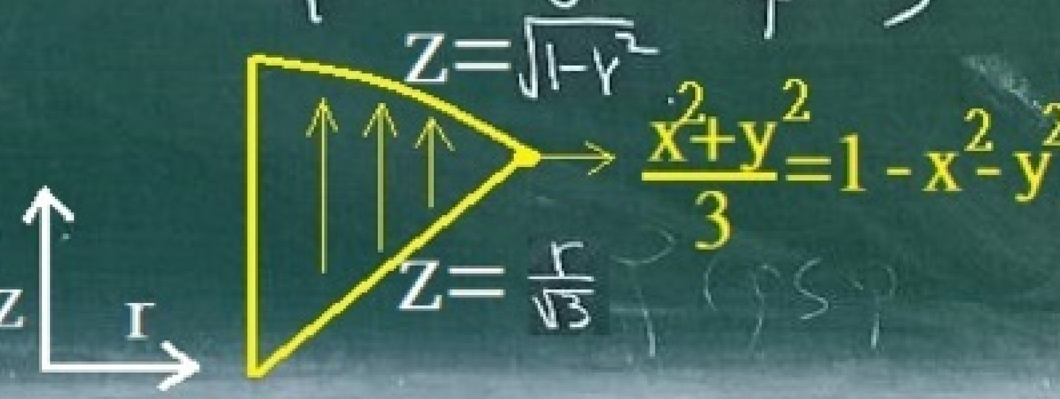
$$R = \left\{ (x,y) \;\exists\, z \in \mathbb{R} \;\; \sqrt{\frac{x^2+y^2}{3}} \le z \le \sqrt{1-(x^2+y^2)} \right\}$$

$$= \left\{ \frac{x^2+y^2}{3} \le 1-(x^2+y^2) \right\}$$

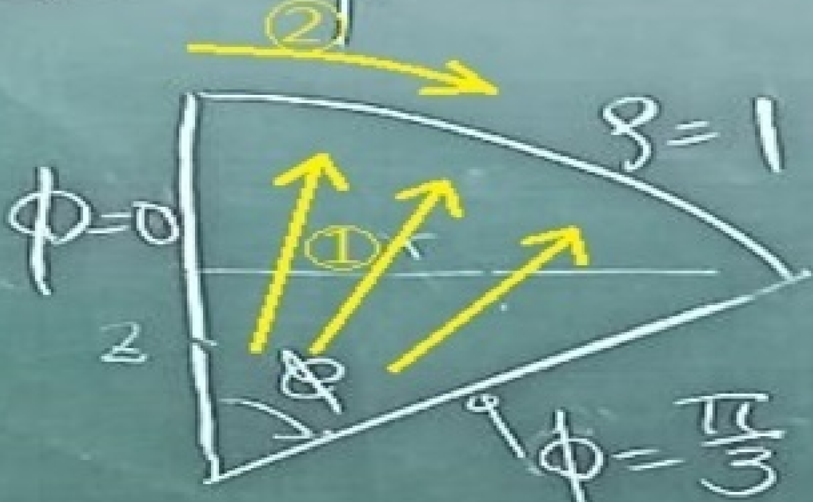
$$= \left\{ x^2+y^2 \le \frac{3}{4} \right\}$$

$$V = \iint_R \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\, dA$$

$$= \int_0^{2\pi}\int_{r=0}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{3}\right) r\, dr\, d\theta$$



II Spherical Coordinate



$(x^2 + y^2 + z^2 = 1)$

$(\tan\phi = \sqrt{3})$

$\left(\frac{r}{z} = \sqrt{3}\right.$, $z = \sqrt{\frac{x^2+y^2}{3}}$

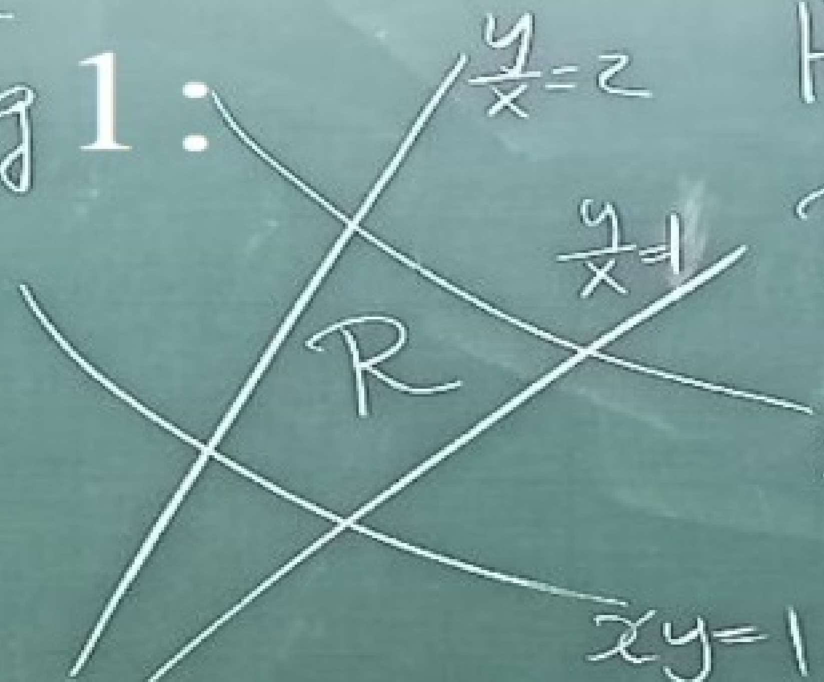
$$D = \left\{ \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \underbrace{\frac{1}{3}}_{d\rho} \underbrace{\left(1 - \frac{1}{2}\right)}_{d\phi} \underbrace{2\pi}_{d\theta} = \frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg1: Find area of



$$R=\left\{\begin{matrix}1\le xy\le 2\\ 1\le \frac{y}{x}\le 2\end{matrix}\right\}=\left\{\begin{matrix}1\le u\le 2\\ 1\le v\le 2\end{matrix}\right\}$$

$$A=\int_{v=1}^{2}\int_{u=1}^{2} dA \qquad \underline{\underline{Q}}:\ dA=\underline{?}\,du\,dv$$

$\underline{\text{Ans}}$: $dA=\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right|du\,dv$

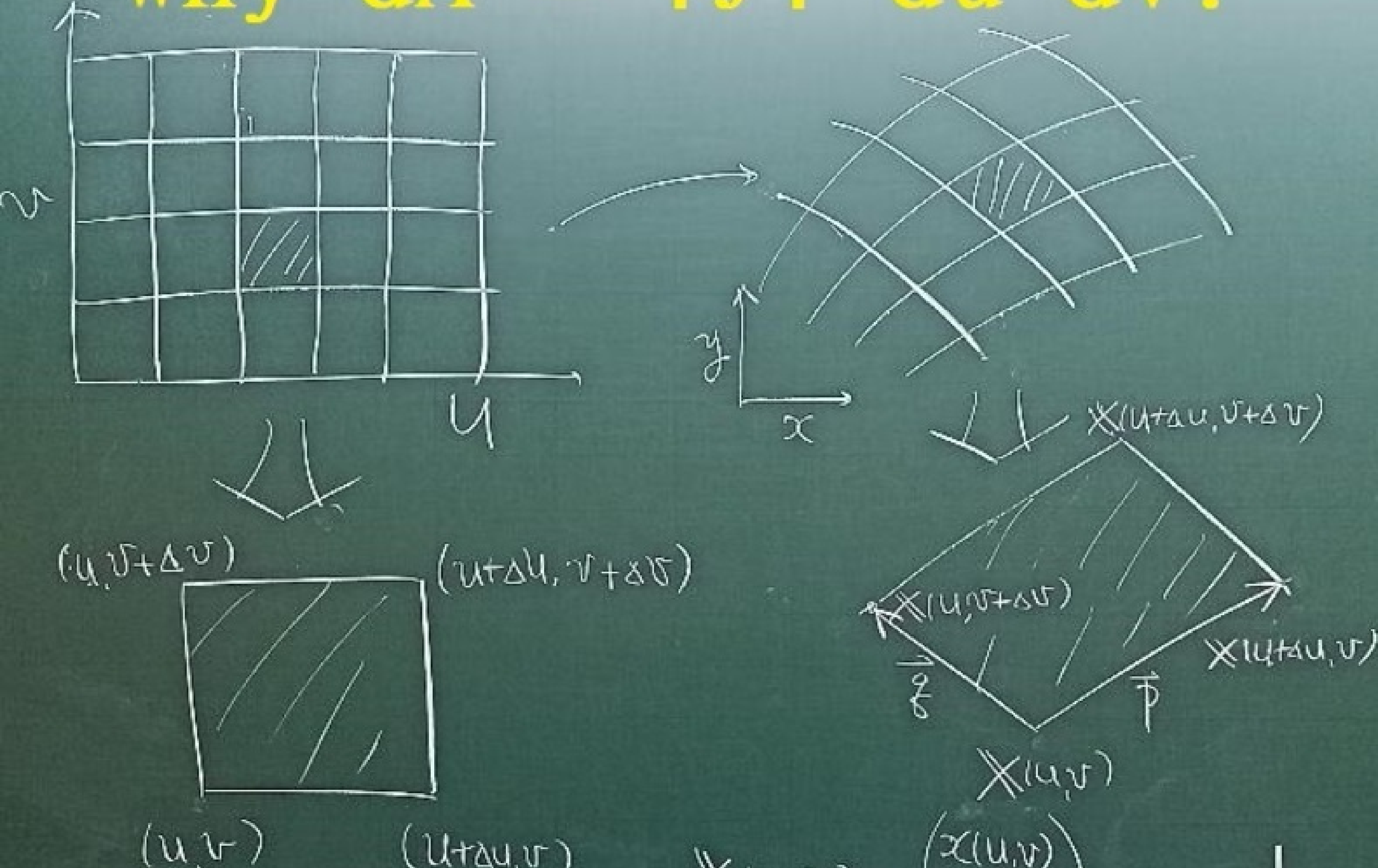
$$\underset{\underset{J}{\shortparallel}}{\frac{\partial(x,y)}{\partial(u,v)}}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$

Need $\begin{cases}u=xy\\ v=\frac{y}{x}\end{cases}$

$\longrightarrow$ $x=x(u,v)=\sqrt{\frac{u}{v}}$

$y=y(u,v)=\sqrt{uv}$

Why dA = |J| du dv:



$$\mathbb{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}, \quad \cdots \text{ etc.}$$

$$\mathbb{X}(u+\Delta u,v) = \begin{pmatrix} x(u+\Delta u,v) \\ y(u+\Delta u,v) \end{pmatrix}$$

$$\Delta A \cong |\vec{p} \times \vec{q}|$$

$$= \left| \left(\mathbb{X}(u+\Delta u) - \mathbb{X}(u) \right) \times \left(\mathbb{X}(u, v+\Delta v) - \mathbb{X}(u,v) \right) \right|$$

$$\underset{=}{\underline{\text{M.V.T.}}} \left| \Delta u \begin{pmatrix} \frac{\partial x}{\partial u}(u + c_{11}\Delta u, v) \\ \frac{\partial y}{\partial u}(u + c_{21}\Delta u, v) \end{pmatrix} \times \Delta v \begin{pmatrix} \frac{\partial x}{\partial v}(u, v + c_{12}\Delta v) \\ \frac{\partial y}{\partial v}(u, v + c_{22}\Delta v) \end{pmatrix} \right|$$

$$= \Delta u \, \Delta v \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \left(0, 0, \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

we sometimes write

$$(a, b) \times (c, d) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \Delta A = |J| \Delta u \Delta v$$

$$J = \det\left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\right)$$

$$= \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eg 1:

$u = xy$

$v = \frac{y}{x}$

$$A = \int_{u=1}^{2}\int_{v=1}^{2} dA$$

$x = \sqrt{\frac{u}{v}}$

$y = \sqrt{uv}$

$$= \int_1^2\int_1^2 |J| \, du \, dv$$

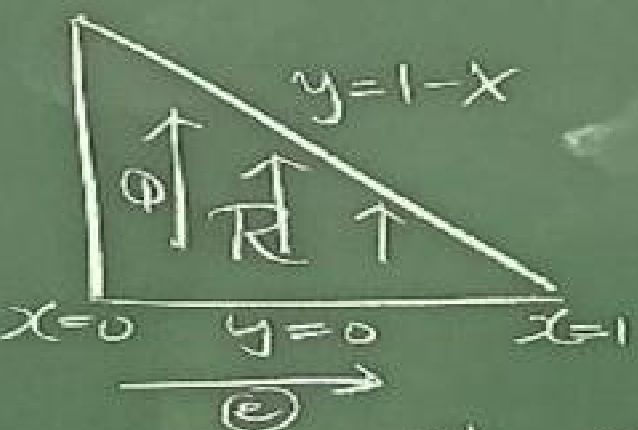
$$= \int_1^2\int_1^2 \left| \det\begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix} \right| du\,dv$$

$$= \int_1^2\int_1^2 \left|\frac{1}{2v}\right| du\,dv = \ln 2/2$$

Eg2 $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x}\underbrace{\sqrt{x+y}}_{\sqrt{u}}\,\underbrace{(x-2y)^2}_{v^2}\,dy\,dx$

$\displaystyle = \int_{?}^{?}\int_{?}^{?}\sqrt{u}\,v^2\,|J|\,du\,dv$

$$\begin{aligned} u &= x+y \\ v &= x-2y \end{aligned} \Rightarrow \begin{aligned} x &= \frac{2u+v}{3} \\ y &= \frac{u-v}{3} \end{aligned} \Rightarrow J = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \frac{-1}{3}$$

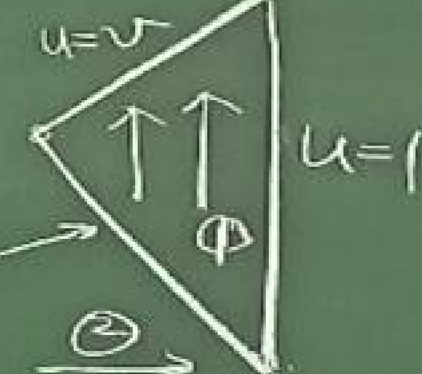


Edges

$y=0$ $u=v$

$y=1-x \Leftrightarrow u=1$

$x=0$ $2u+v=0$



$\displaystyle \text{Ans} = \int_{u=0}^{1}\int_{v=-2u}^{u}\sqrt{u}\cdot v^2\cdot\left|\frac{-1}{3}\right|dv\,du$

$\displaystyle = \int_{0}^{1}\sqrt{u}\left(\int_{v=-2u}^{u}\frac{v^2}{3}\,dv\right)du$

$\displaystyle = \int_{0}^{1}u^{\frac{1}{2}}\left(\frac{v^3}{9}\Big|_{-2u}^{u}\right)du = \int_{0}^{1}u^{\frac{7}{2}}\,du = \frac{2}{9}$