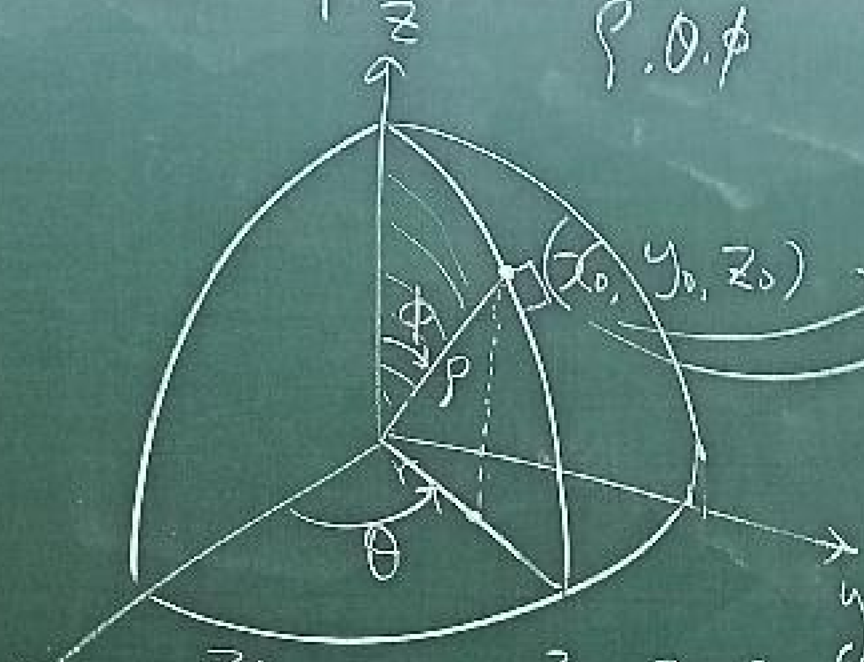
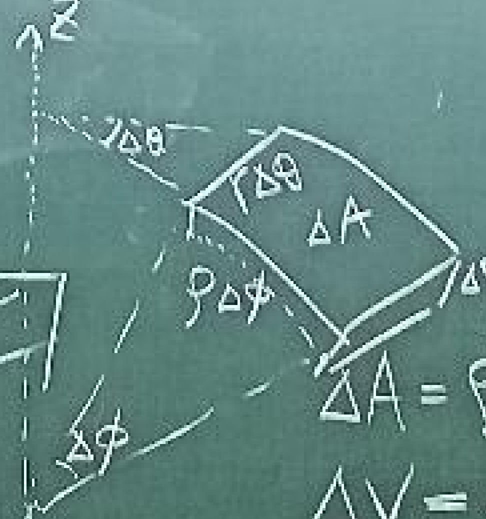
II Spherical Coordinates

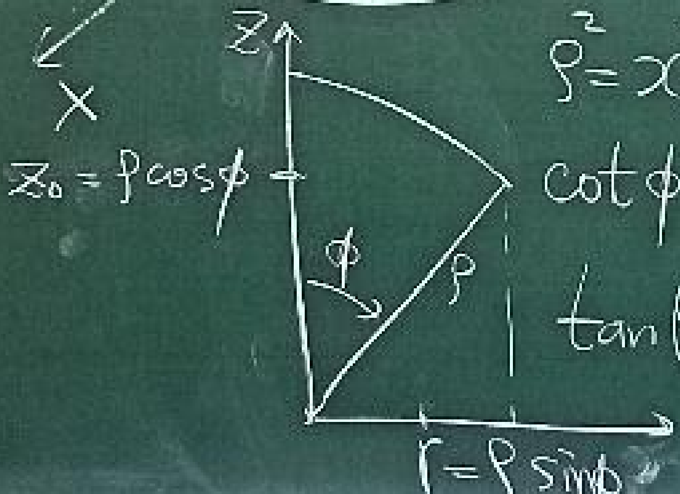
ρ, θ, ϕ





$$\Delta A = \rho^2 \sin\phi \, \Delta\theta \, \Delta\phi$$

$$\Delta V = \Delta A \, \Delta\rho = \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$$



$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot\phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan\theta = \frac{y_0}{x_0}$$

$$x_0 = r\cos\theta = \rho\sin\phi\cos\theta$$

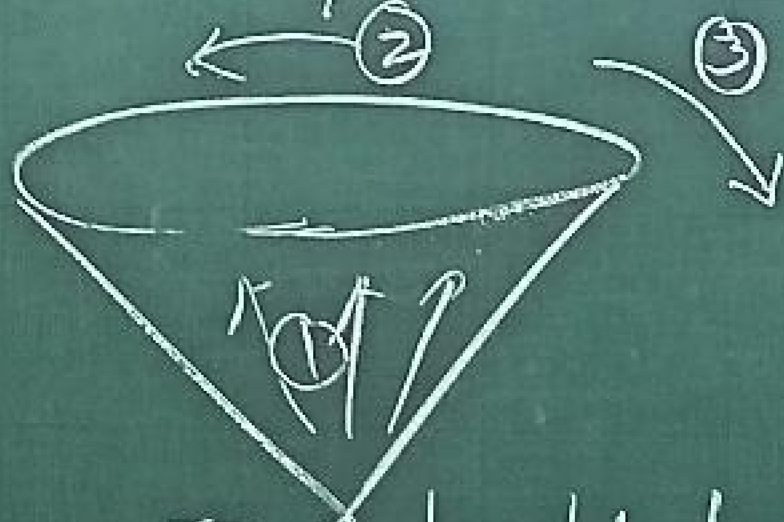
$$y_0 = r\sin\theta = \rho\sin\phi\sin\theta$$

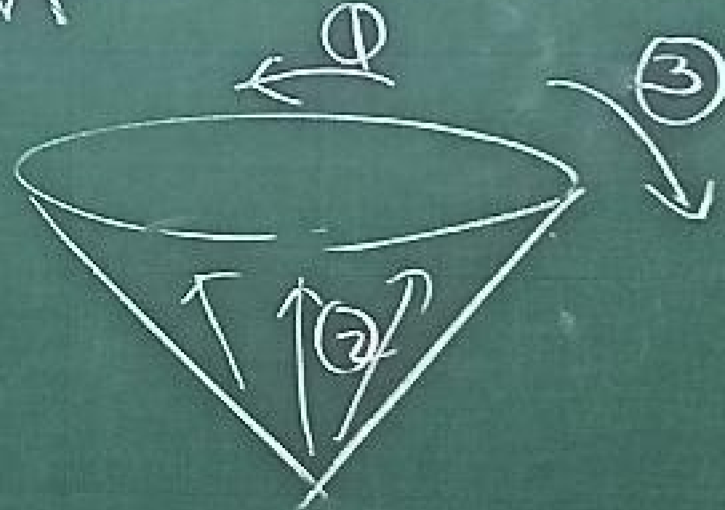
$$z_0 = \rho\cos\phi$$

Cross Sections for various order of integration

Case I: $d\rho\, d\theta\, d\phi$ or $d\theta\, d\rho\, d\phi$

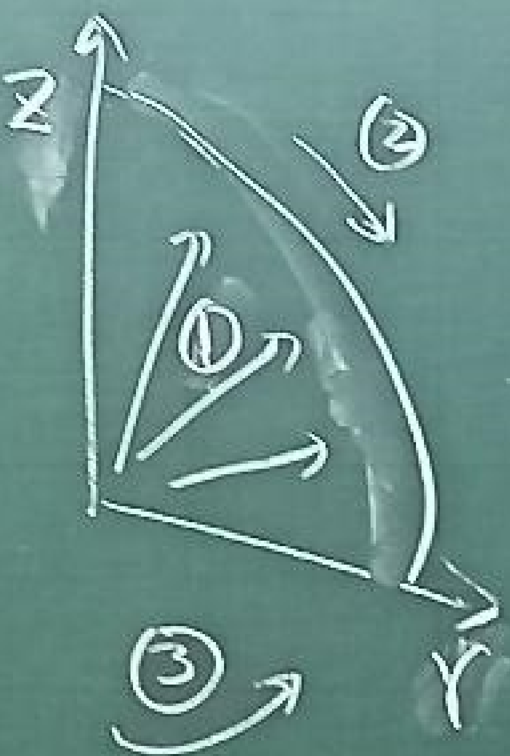
$\phi = \text{constant}$

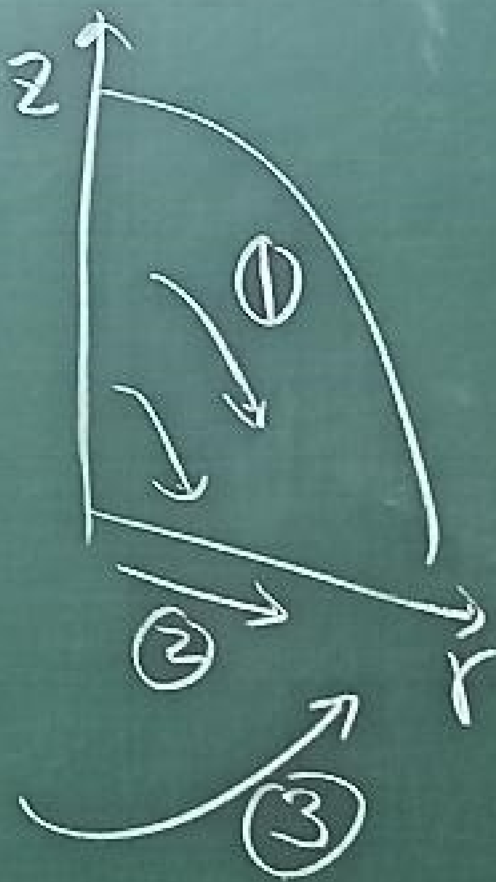




case II $d\theta\, d\phi\, d\rho$ $d\phi\, d\theta\, d\rho$

Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$

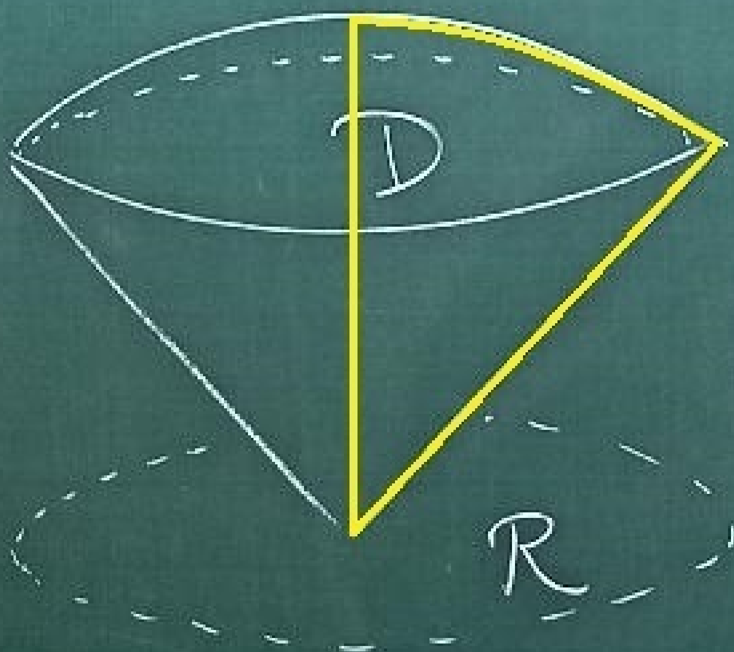




Easiest case in general
Must learn!

Example: $D=\left\{\begin{array}{l} x^2+y^2+z^2\le 1 \\ z\ge\sqrt{\dfrac{x^2+y^2}{3}}\end{array}\right\}$

Find volume of D



$\underline{\text{Sol}}$ (I). Cylindrical coordinate

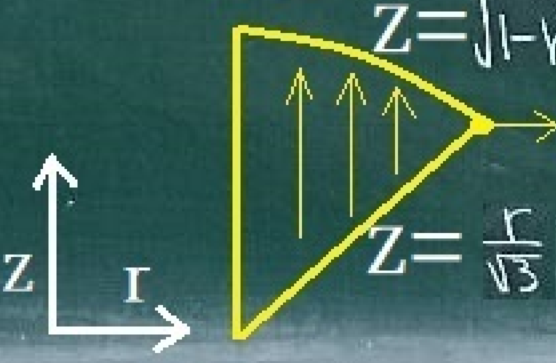
$$R=\left\{\begin{array}{l}(x,y)\ \exists z\in\mathbb{R} \\ \sqrt{\dfrac{x^2+y^2}{3}}\le z\le\sqrt{1-(x^2+y^2)}\end{array}\right\}$$

$$=\left\{\frac{x^2+y^2}{3}\le 1-(x^2+y^2)\right\}$$

$$=\left\{x^2+y^2\le\frac{3}{4}\right\}$$

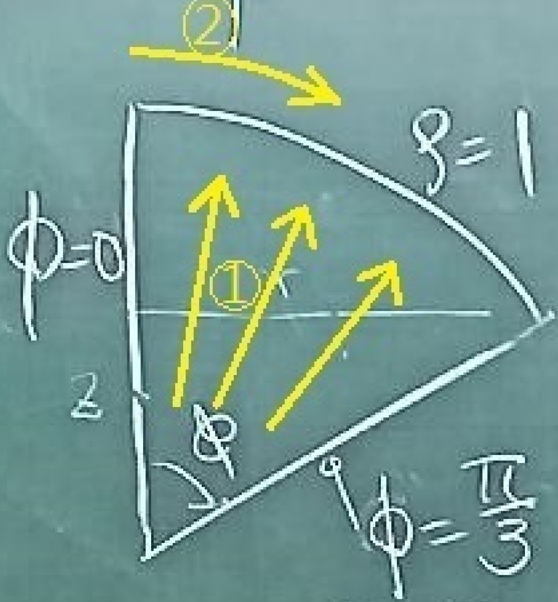
$$V=\iint_R\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\,dA$$

$$=\int_0^{2\pi}\int_{r=0}^{\frac{\sqrt{3}}{2}}\left(\sqrt{1-r^2}-\frac{r}{3}\right)r\,dr\,d\theta$$



$\dfrac{x^2+y^2}{3}=1-x^2-y^2$

II Spherical Coordinate



$(x^2+y^2+z^2=1)$

$$\left(\frac{r}{z}=\sqrt{3},\ \tan\phi=\sqrt{3}\right)$$

$$z=\sqrt{\frac{x^2+y^2}{3}}$$

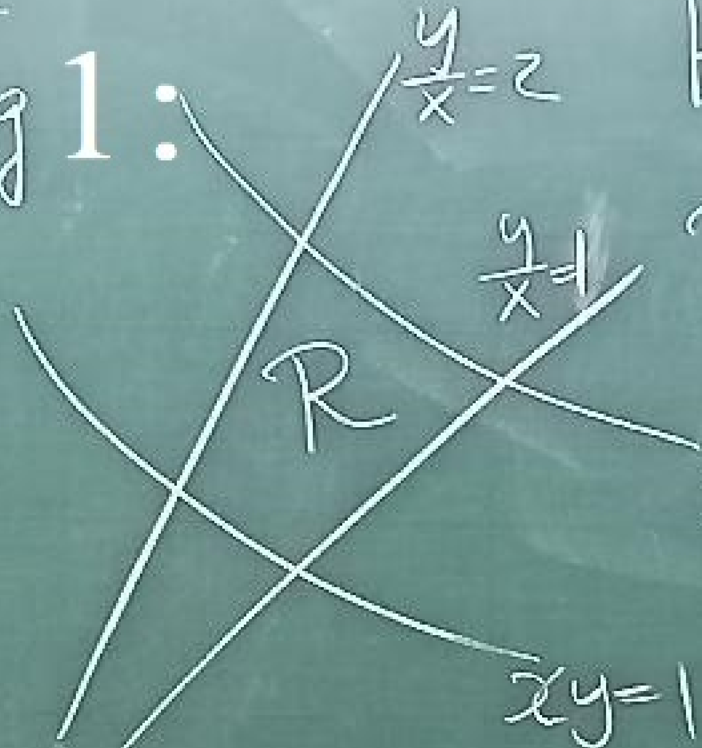
$$D=\left\{\begin{array}{l} 0\le\rho\le 1 \\ 0\le\phi\le\frac{\pi}{3} \\ 0\le\theta\le 2\pi \end{array}\right\}$$

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\underbrace{\frac{1}{3}}_{d\rho}\underbrace{\left(1-\frac{1}{2}\right)}_{d\phi}\underbrace{2\pi}_{d\theta}=\frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg1: Find area of



$$R=\left\{\begin{array}{l}1\le xy\le 2\\ 1\le \frac{y}{x}\le 2\end{array}\right\}=\left\{\begin{array}{l}1\le u\le 2\\ 1\le v\le 2\end{array}\right\}$$

$$A=\int_{v=1}^{2}\int_{u=1}^{2} dA$$

$\underline{\underline{Q}}$: $dA=\underline{?}\,du\,dv$

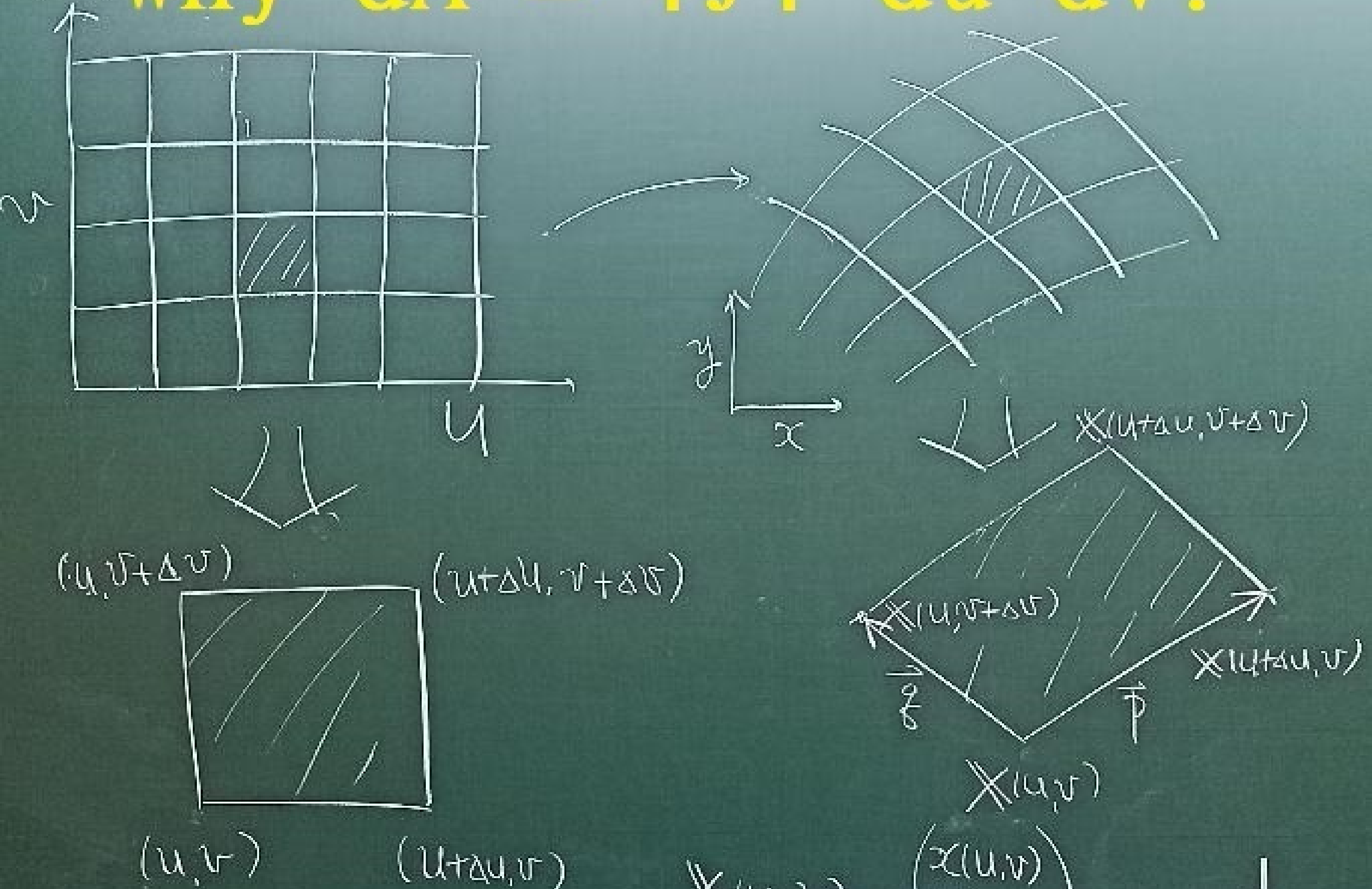
$\underline{\text{Ans}}$: $dA=\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right|du\,dv$

$$\underset{\substack{\parallel\\ J}}{\frac{\partial(x,y)}{\partial(u,v)}}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$

Need $\left\{\begin{array}{l}u=xy\\ v=\frac{y}{x}\end{array}\right.$ $\longrightarrow$ $x=x(u,v)=\sqrt{\frac{u}{v}}$

$y=y(u,v)=\sqrt{uv}$

Why dA = |J| du dv:



$$\mathbb{X}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}, \quad \cdots \text{ etc.}$$

$$\mathbb{X}(u+\Delta u,v) = \begin{pmatrix} x(u+\Delta u,v) \\ y(u+\Delta u,v) \end{pmatrix}$$

$$\Delta A \cong \left| \vec{p} \times \vec{q} \right|$$

$$= \left| \left(\mathbb{X}(u+\Delta u) - \mathbb{X}(u) \right) \times \left(\mathbb{X}(u, v+\Delta v) - \mathbb{X}(u,v) \right) \right|$$

$$\underset{\text{M.V.T.}}{=} \left| \Delta u \begin{pmatrix} \frac{\partial x}{\partial u}(u + c_{11}\Delta u, v) \\ \frac{\partial y}{\partial u}(u + c_{21}\Delta u, v) \end{pmatrix} \times \Delta v \begin{pmatrix} \frac{\partial x}{\partial v}(u, v + c_{12}\Delta v) \\ \frac{\partial y}{\partial v}(u, v + c_{22}\Delta v) \end{pmatrix} \right|$$

$$= \Delta u \Delta v \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \left(0, 0, \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

We sometimes write

$$(a, b) \times (c, d) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\therefore \Delta A = |J| \Delta u \Delta v$

$$J = \det\begin{pmatrix} \cancel{\frac{\partial x}{\partial u}} & \cancel{\frac{\partial x}{\partial v}} \end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eg 1:

$$A = \int_{u=1}^{2}\int_{v=1}^{2} dA \qquad u = xy, \quad v = \frac{y}{x}$$

$$= \int_1^2\int_1^2 |J|\, du\, dv \qquad x = \sqrt{\frac{u}{v}}, \quad y = \sqrt{uv}$$

$$= \int_1^2\int_1^2 \left| \det\begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{uv}} & \frac{-1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{pmatrix} \right| du\, dv$$

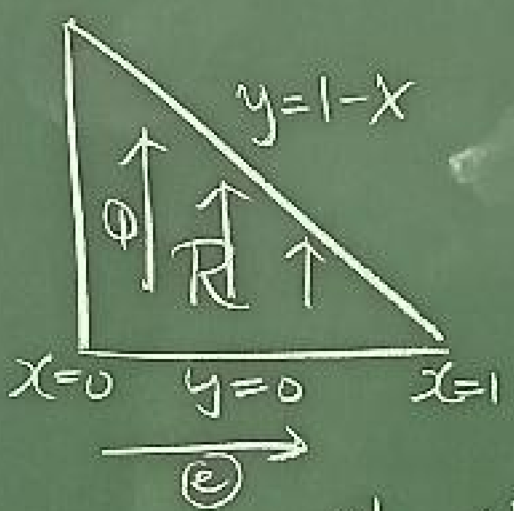
$$= \int_1^2\int_1^2 \left|\frac{1}{2v}\right| du\, dv = \ln 2/2$$

Eg 2
$$\int_{x=0}^{1}\int_{y=0}^{1-x}\underbrace{\sqrt{x+y}}_{\sqrt{u}}\underbrace{(x-2y)^2}_{v^2}\,dy\,dx$$

$$=\int_{?}^{?}\int_{?}^{?}\sqrt{u}\,v^2\,|J|\,du\,dv$$

$$\begin{aligned} u&=x+y \\ v&=x-2y \end{aligned} \Rightarrow \begin{aligned} x&=\frac{2u+v}{3} \\ y&=\frac{u-v}{3} \end{aligned} \Rightarrow J=\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix}=\frac{-1}{3}$$

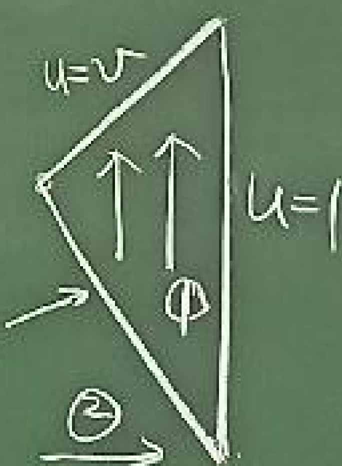
Edges



$y=0 \iff u=v$

$y=1-x \iff u=1$

$x=0 \iff 2u+v=0$



$$\text{Ans}=\int_{u=0}^{1}\int_{v=-2u}^{u}\sqrt{u}\cdot v^2\cdot\left|\frac{-1}{3}\right|dv\,du$$

$$=\int_0^1\sqrt{u}\left(\int_{v=-2u}^{u}\frac{v^2}{3}dv\right)du$$

$$=\int_0^1 u^{\frac{1}{2}}\left(\frac{v^3}{9}\Big|_{-2u}^{u}\right)du=\int_0^1 u^{\frac{7}{2}}du=\frac{2}{9}$$