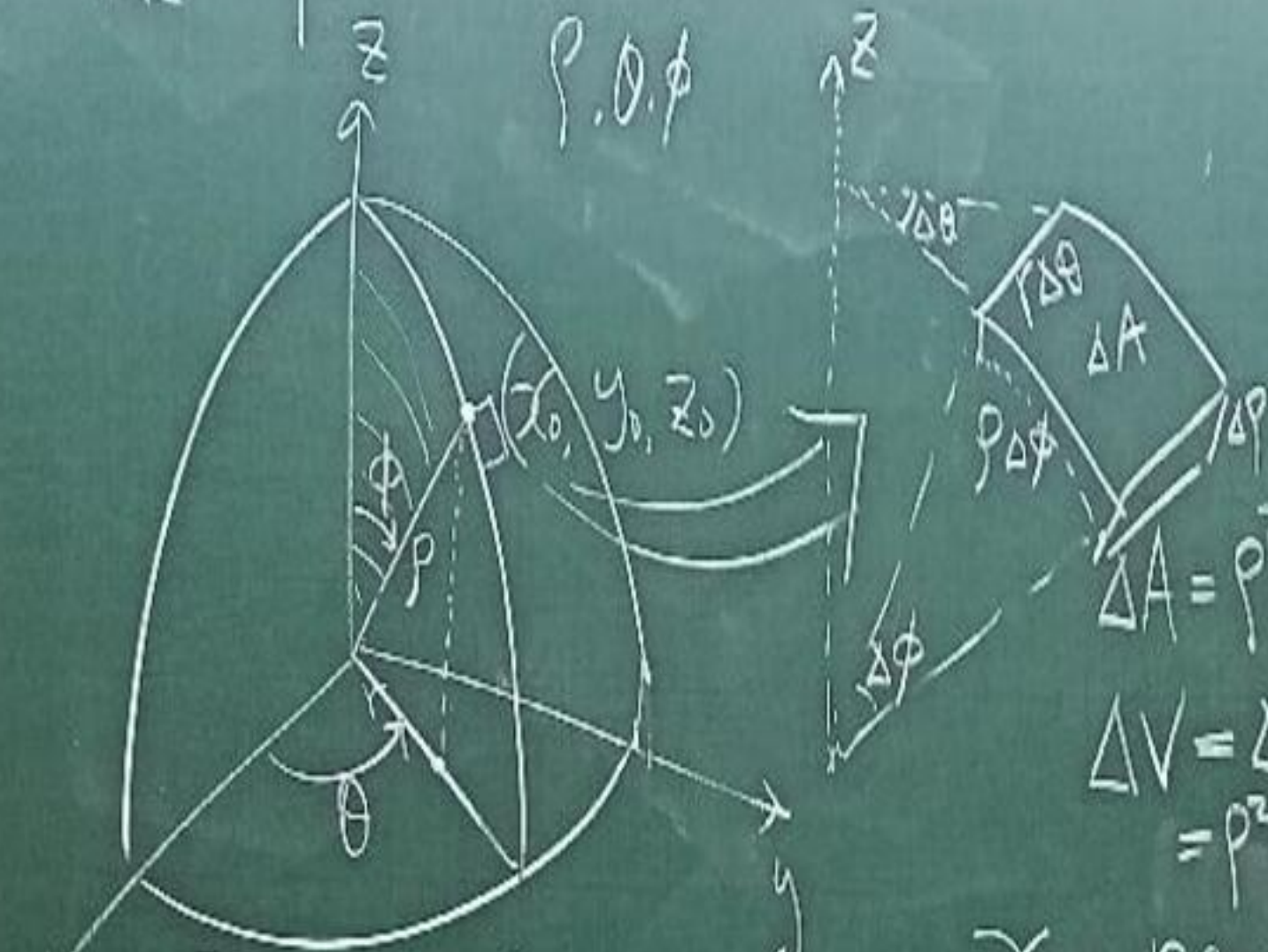


II Spherical Coordinates



$$\Delta A = \rho^2 \sin \phi \Delta \theta \Delta \phi$$

$$\Delta V = \Delta A \Delta \rho$$

$$= \rho^2 \sin \phi \Delta \rho \Delta \theta \Delta \phi$$

$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot \phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan \theta = \frac{y_0}{x_0}$$

$$r = \rho \sin \phi$$

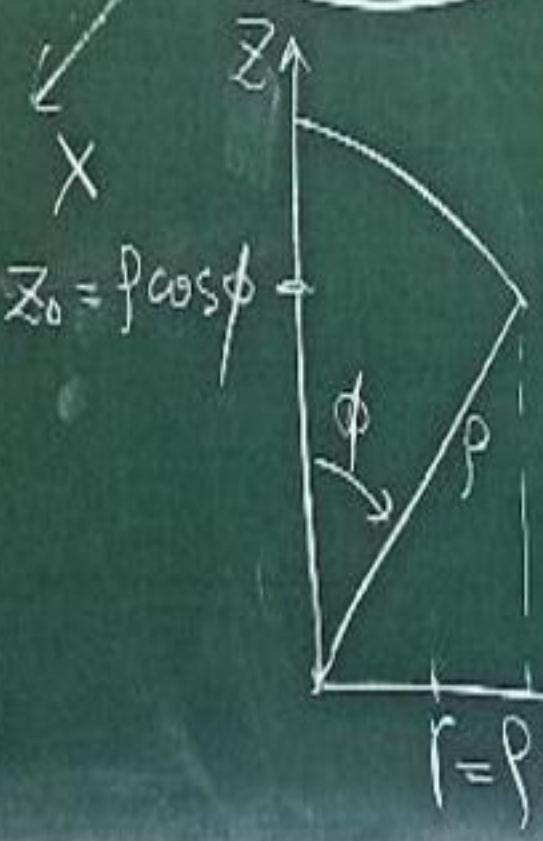
$$x_0 = r \cos \theta$$

$$= \rho \sin \phi \cos \theta$$

$$y_0 = r \sin \theta$$

$$= \rho \sin \phi \sin \theta$$

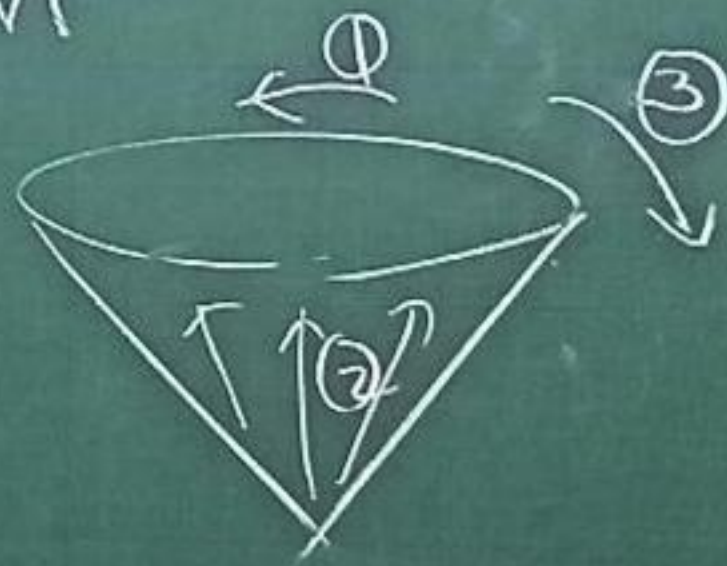
$$z_0 = \rho \cos \phi$$



Cross Sections for various order of integration

Case I: $dy d\theta d\phi$ or $d\theta dy d\phi$

$\phi = \text{constant}$



Case II: $d\theta d\phi dy$



$d\phi d\theta dy$

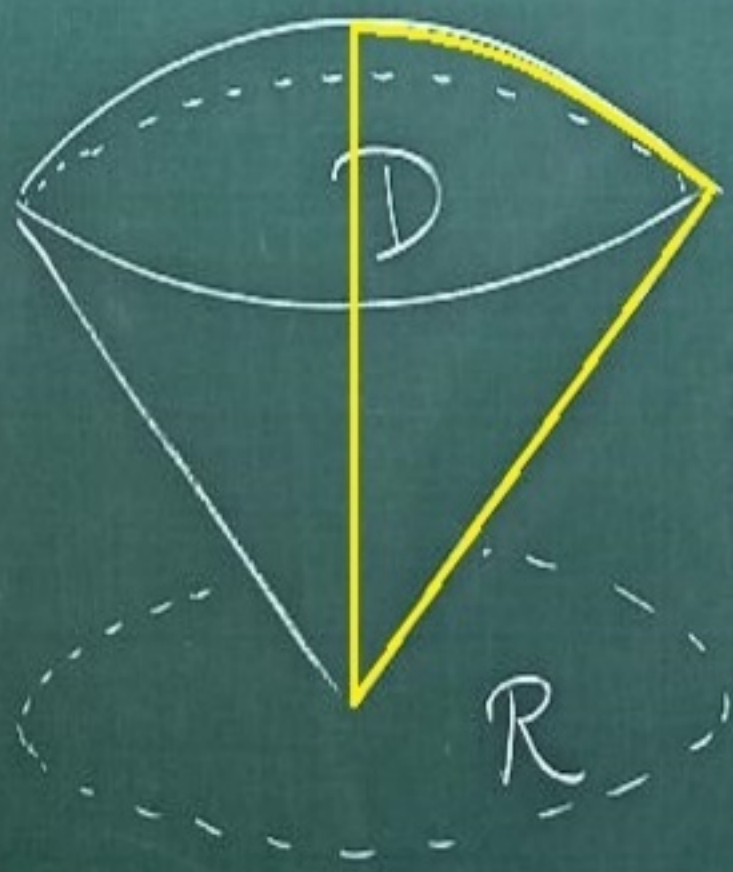


Case III $d\rho d\phi d\theta$, $d\phi d\rho d\theta$



Easiest case in general
Must learn!

Example. $D = \left\{ \begin{array}{l} x^2 + y^2 + z^2 \leq 1 \\ z \geq \sqrt{\frac{x^2 + y^2}{3}} \end{array} \right\}$
 Find volume of D



Sol (I). Cylindrical coordinate.

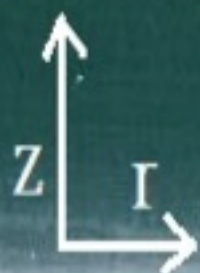
$$R = \left\{ \begin{array}{l} (x, y) \exists z \in \mathbb{R} \\ \sqrt{\frac{x^2 + y^2}{3}} \leq z \leq \sqrt{1 - (x^2 + y^2)} \end{array} \right\}$$

$$= \left\{ \frac{x^2 + y^2}{3} \leq 1 - (x^2 + y^2) \right\}$$

$$= \left\{ x^2 + y^2 \leq \frac{3}{4} \right\}$$

$$V = \iint_R \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz dA$$

$$= \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}} \right) r dr d\theta$$



II Spherical Coordinate



$$\left(\frac{x}{z} = \sqrt{3}, \tan \phi = \sqrt{3} \right)$$

$$z = \sqrt{\frac{x^2 + y^2}{3}}$$

$$D = \left\{ \begin{array}{l} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \frac{\pi}{3} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left(\frac{1}{3} \left(1 - \frac{1}{2} \right) \right) d\phi \, d\theta = \frac{\pi}{3}$$

Substitution in Multiple Integrals

Eg 1: $\frac{y}{x}=2$ Find area of $R = \left\{ \begin{array}{l} 1 \leq xy \leq 2 \\ 1 \leq \frac{y}{x} \leq 2 \end{array} \right\}$

$xy=2 = \left\{ \begin{array}{l} 1 \leq u \leq 2 \\ 1 \leq v \leq 2 \end{array} \right\}$

$xy=1$

$$A = \int_{x=1}^2 \int_{u=1}^2 dA, \quad \underline{Q}: dA = ? du dv$$

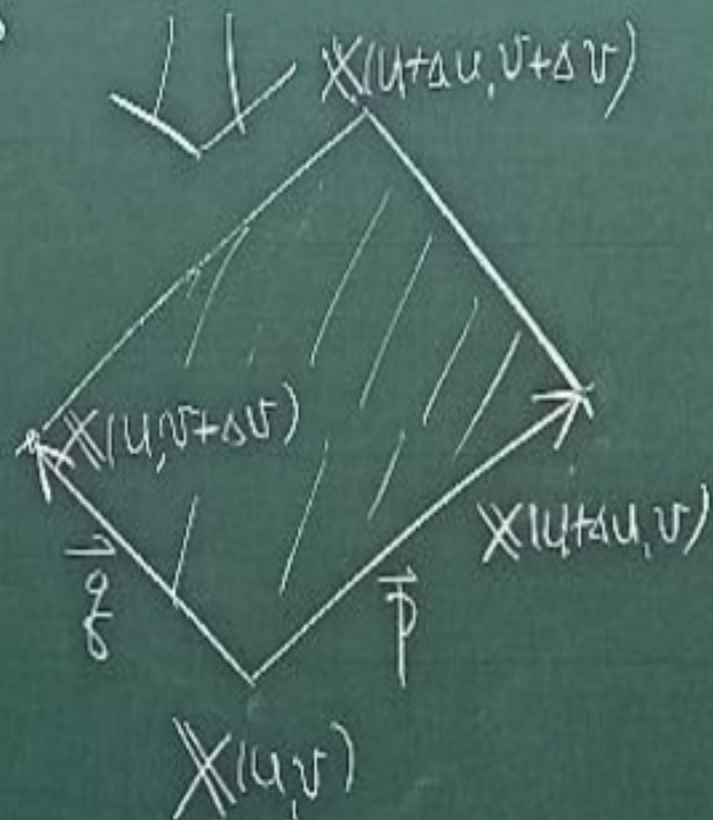
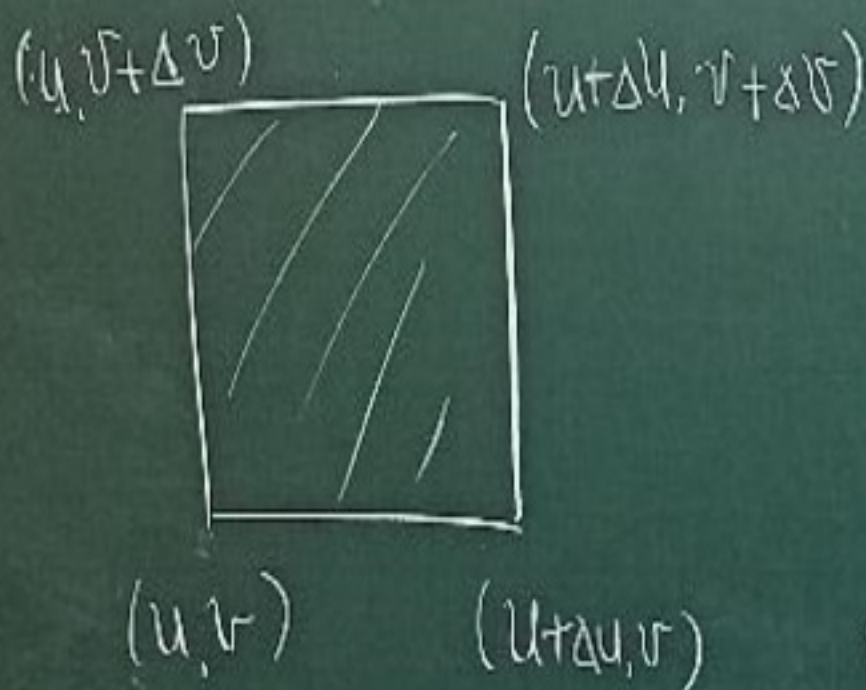
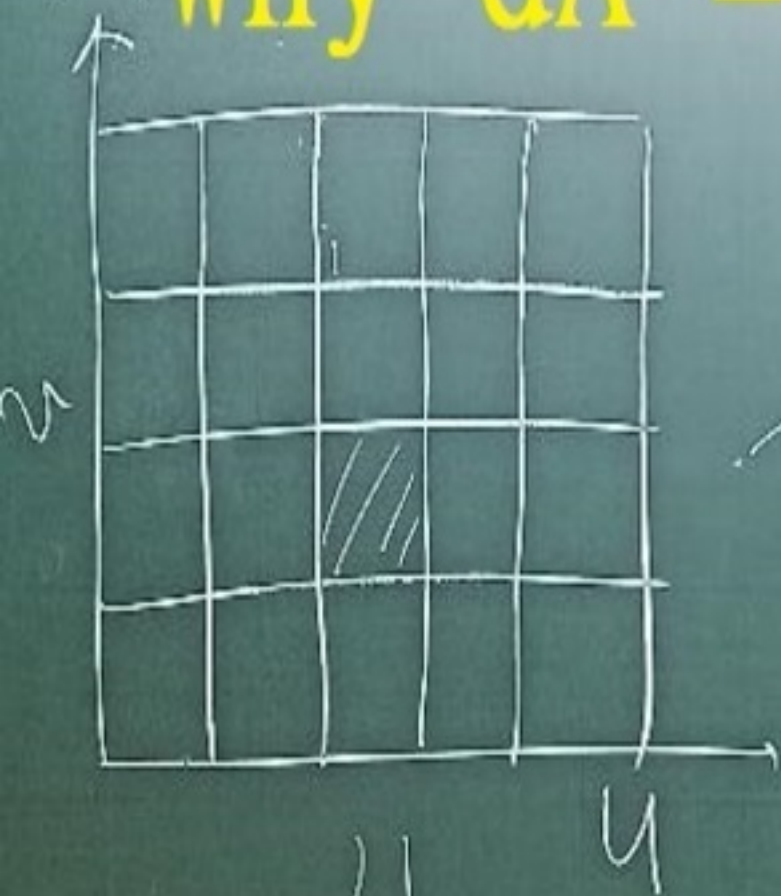
Ans: $dA = \left| \det \left(\frac{\partial(x,y)}{\partial(u,v)} \right) \right| du dv$

$$\frac{\partial(x,y)}{\partial(u,v)} \stackrel{J}{=} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Need $\begin{cases} u=xy \\ v=\frac{y}{x} \end{cases}$

$$\rightarrow \begin{cases} x = x(u,v) = \sqrt{\frac{u}{v}} \\ y = y(u,v) = \sqrt{uv} \end{cases}$$

Why $dA = |J| du dv$:



$$X(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ \dots \text{etc.} \end{pmatrix}$$

$$X(u+\Delta u, v) = \begin{pmatrix} x(u+\Delta u, v) \\ y(u+\Delta u, v) \\ \dots \end{pmatrix}$$

$$\Delta A \approx |\vec{P} \times \vec{q}|$$

$$= \left| \left(X(u+\Delta u) - X(u) \right) \times \left(X(u, v+\Delta v) - X(u, v) \right) \right|$$

$$\stackrel{\text{M.V.T.}}{=} \left| \Delta u \begin{pmatrix} \frac{\partial x}{\partial u}(u + C_{11}\Delta u, v) \\ \frac{\partial y}{\partial u}(u + C_{21}\Delta u, v) \end{pmatrix} \times \Delta v \begin{pmatrix} \frac{\partial x}{\partial v}(u, v + C_{12}\Delta v) \\ \frac{\partial y}{\partial v}(u, v + C_{22}\Delta v) \end{pmatrix} \right|$$

$$= \Delta u \Delta v \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = (0, 0, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

We sometimes write

$$(a, b) \times (c, d) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \Delta A = |J| \Delta u \Delta v$$

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eq 1:

$$A = \int_{u=1}^2 \int_{v=1}^2 dA$$

$$u = xy$$

$$v = \frac{y}{x}$$

$$= \int_1^2 \int_1^2 |J| du dv$$

$$x = \sqrt{\frac{u}{v}}$$

$$y = \sqrt{uv}$$

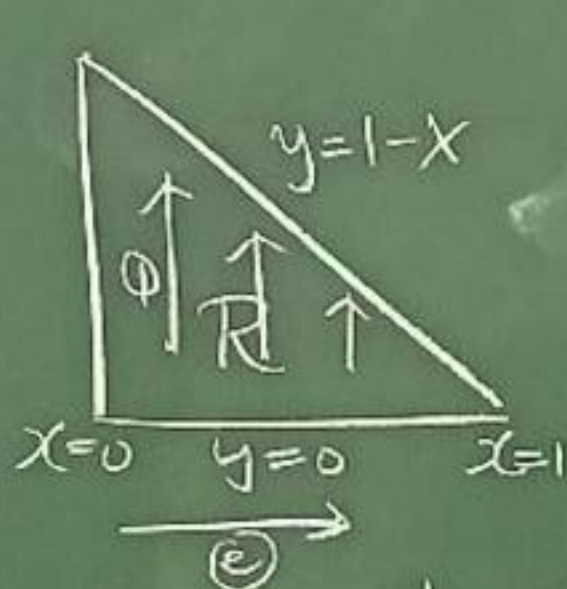
$$= \int_1^2 \int_1^2 \left| \det \begin{pmatrix} \frac{1}{2} \frac{1}{\sqrt{uv}} & -\frac{1}{2} \sqrt{\frac{u}{v^3}} \\ \frac{1}{2} \sqrt{\frac{v}{u}} & \frac{1}{2} \sqrt{\frac{u}{v}} \end{pmatrix} \right| du dv$$

$$= \int_1^2 \int_1^2 \frac{1}{2v} du dv = \ln 2 / 2$$

$$\text{[eg 2]} \int_{x=0}^1 \int_{y=0}^{1-x} \sqrt{x+y} (x-2y)^2 dy dx$$

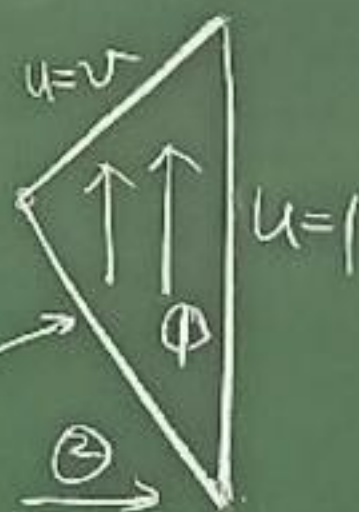
$$= \int_{?}^{?} \int_{?}^{?} \sqrt{u} v^2 |J| du dv$$

$$\begin{aligned} u &= x+y \\ v &= x-2y \end{aligned} \Rightarrow \begin{aligned} x &= \frac{2u+v}{3} \\ y &= \frac{u-v}{3} \end{aligned} \Rightarrow J = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = \frac{-1}{3}$$



Edges

$$\begin{aligned} y=0 & \Leftrightarrow u=v \\ y=1-x & \Leftrightarrow u=1 \\ x=0 & \Leftrightarrow 2u+v=0 \end{aligned}$$



$$\text{Ans} = \int_{u=0}^1 \int_{v=-2u}^u \sqrt{u} \cdot v^2 \cdot \left| \frac{-1}{3} \right| dv du$$

$$= \int_0^1 \sqrt{u} \left(\int_{v=-2u}^u \frac{v^2}{3} dv \right) du$$

$$= \int_0^1 u^{\frac{1}{2}} \left(\frac{v^3}{9} \Big|_{-2u}^u \right) du = \int_0^1 u^{\frac{7}{2}} du = \frac{2}{9}$$