Triple Integrals in Cylindrical Coord.

Cylindrical coordinate
= polar coordinate + z coordinate

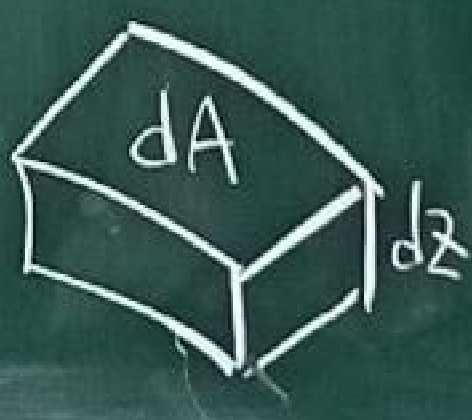
$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$z = z \qquad z = z$$

Here $\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) & (x,y) \in \mathrm{I}, \mathrm{IV} \\ \tan^{-1}\left(\frac{y}{x}\right) \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \; \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right) & (x,y) \in \mathrm{II}, \mathrm{III} \end{cases}$

$$dV = dA\, dz = r\, dr\, d\theta\, dz$$

(or other ordering)



Triple Integrals in Cylindrical and Spherical Coordinates

I. Cylindrical Coordinate

= polar coordinate + z coord.

$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$$

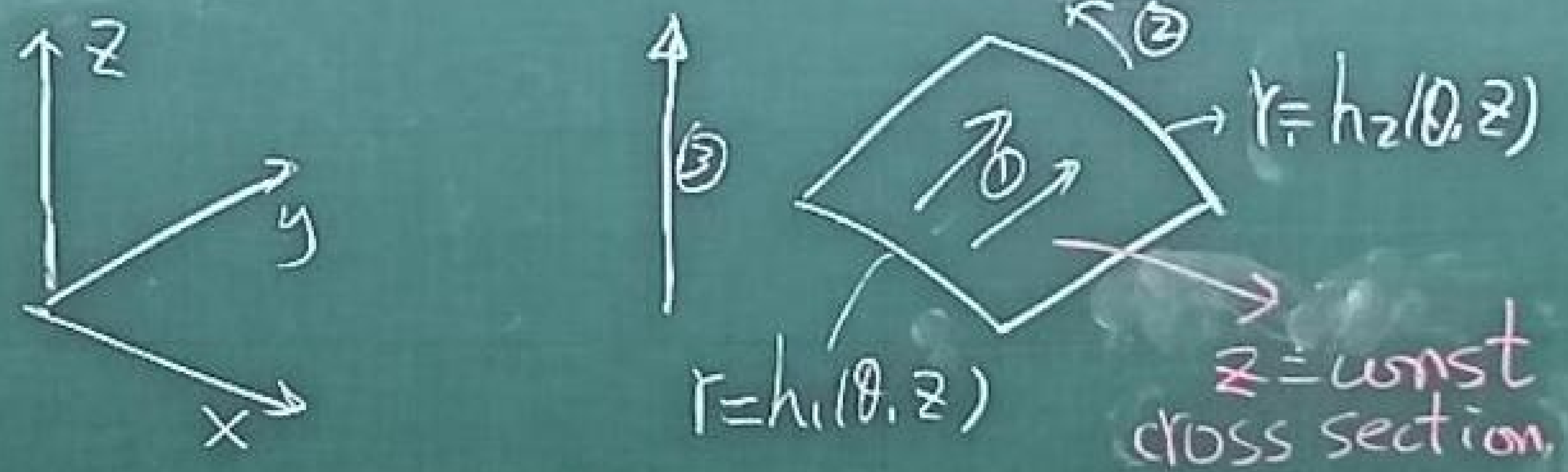
$$r = \sqrt{x^2+y^2},\quad \tan\theta = \frac{y}{x},\quad z = z$$

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \mathrm{I}, \mathrm{IV} \\ \tan^{-1}\frac{y}{x} \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),\ \left(\frac{-3\pi}{2}, \frac{\pi}{2}\right) & (x,y) \in \frac{\mathrm{II}}{\mathrm{III}} \end{cases}$$

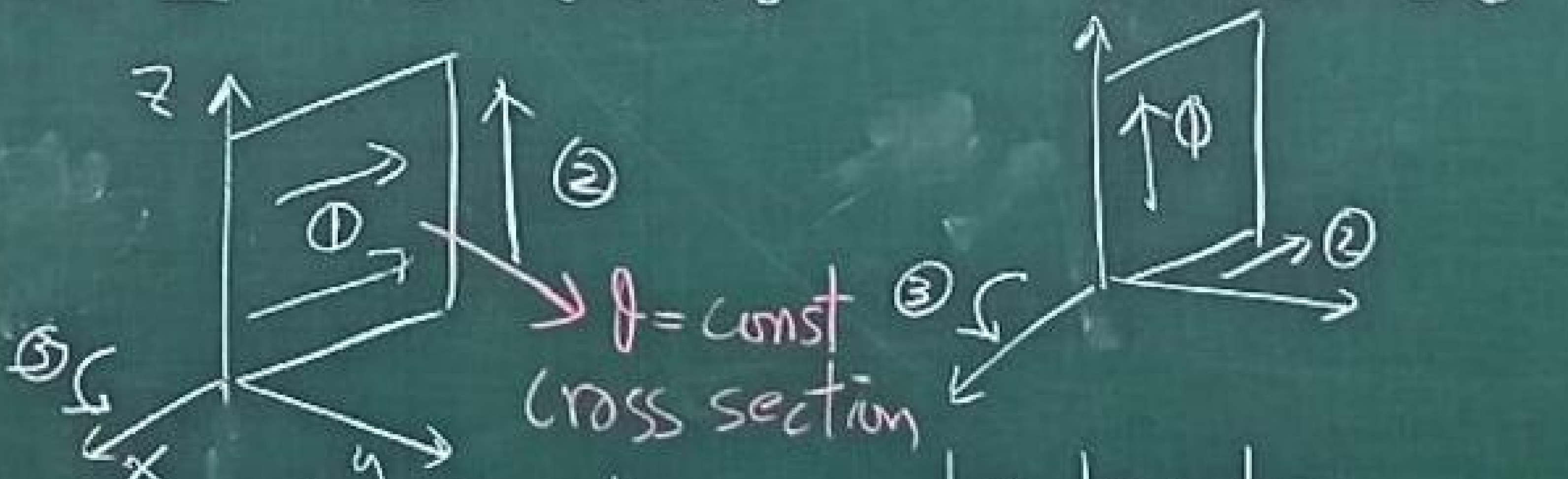
$$dV = dA \cdot dz = r\,dr\,d\theta\,dz$$

Finding limits of integration

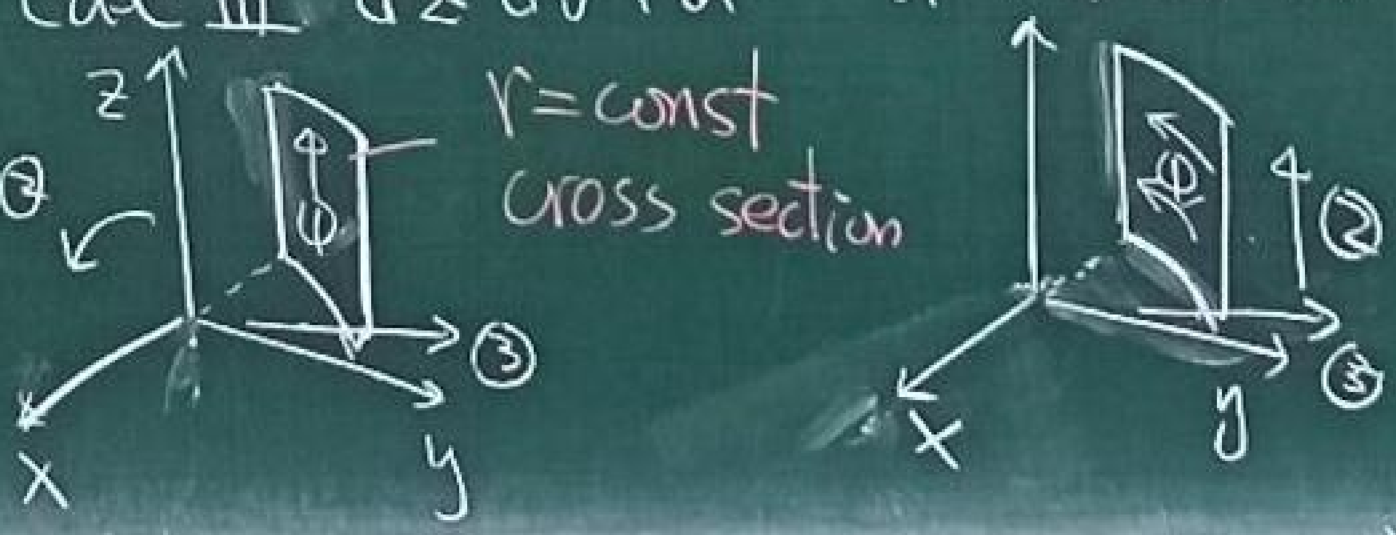
Case I: $r\,dr\,d\theta\,dz$ ($d\theta\,r\,dr\,dz$ is similar)



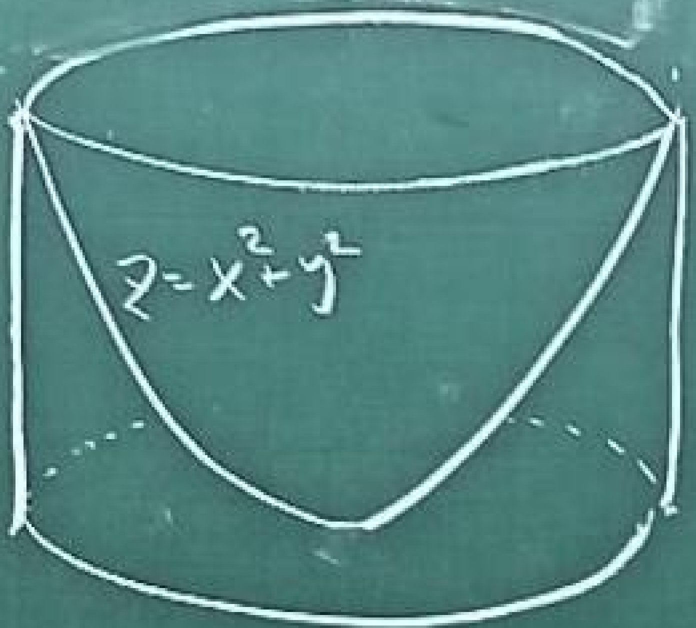
Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$



Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$



Eg $D = \left\{ \begin{array}{l} x^2+y^2 \le 4 \\ 0 \le z \le x^2+y^2 \end{array} \right\}$



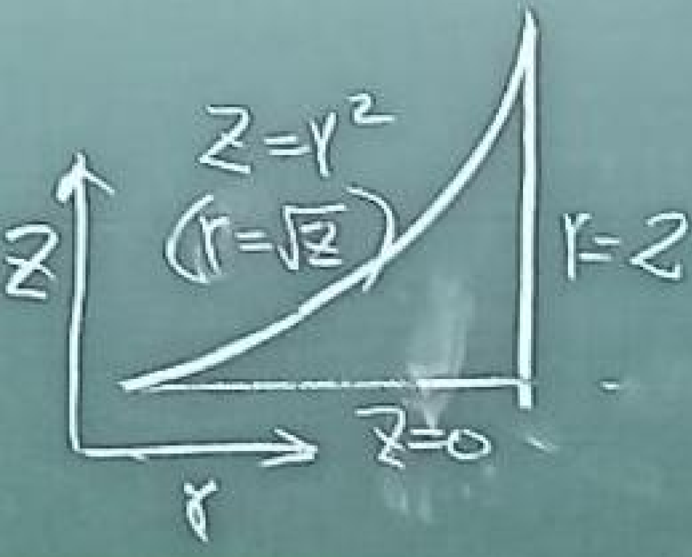
Find volume of D using Cylindrical coordinates.

case I: $r\,dr\,d\theta\,dz$

Step 1 cross section

$$V = \int_0^4 \int_0^{2\pi} \int_{r=\sqrt{z}}^{2} r\,dr\,d\theta\,dz$$

Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

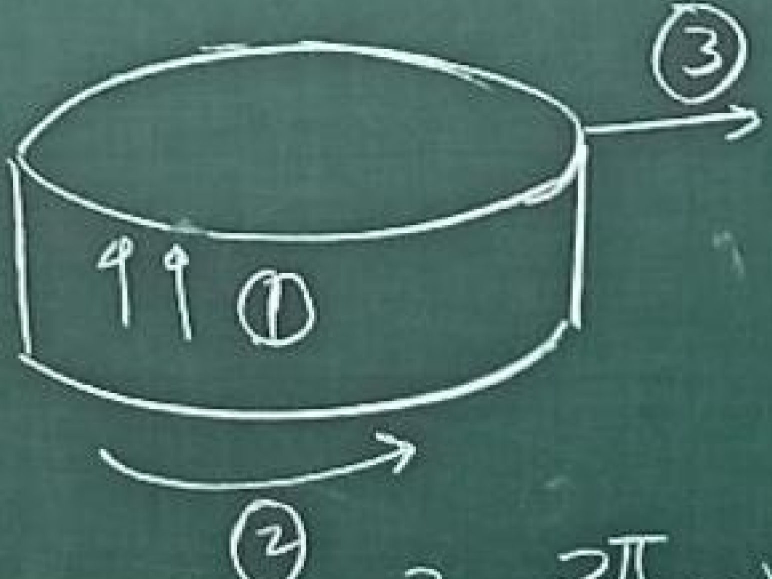


$$V=\int_0^{2\pi}\int_0^{4}\int_{r=\sqrt{z}}^{2} r\,dr\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2}\int_{z=0}^{r^2} dz\,r\,dr\,d\theta$$

$$=8\pi$$

Case III $dz\,d\theta\,r\,dr$

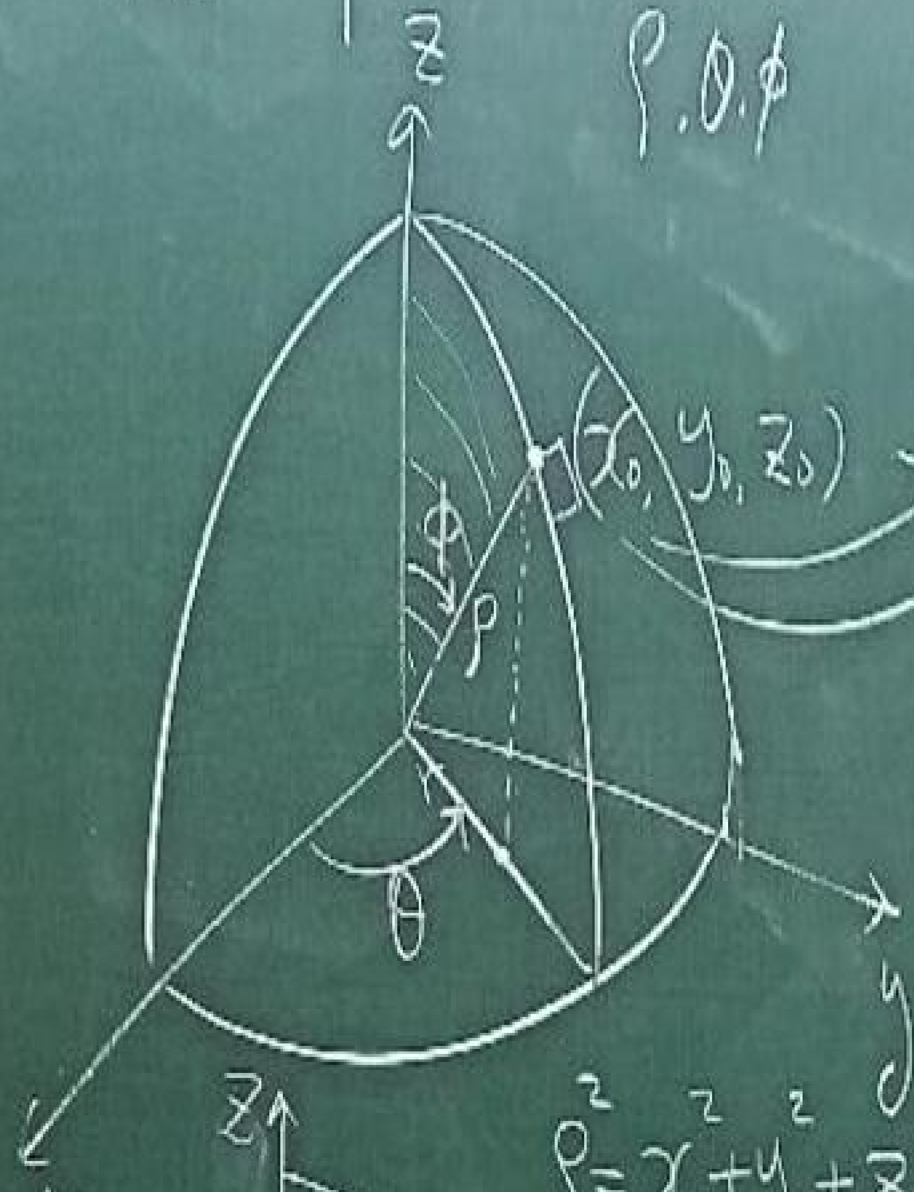


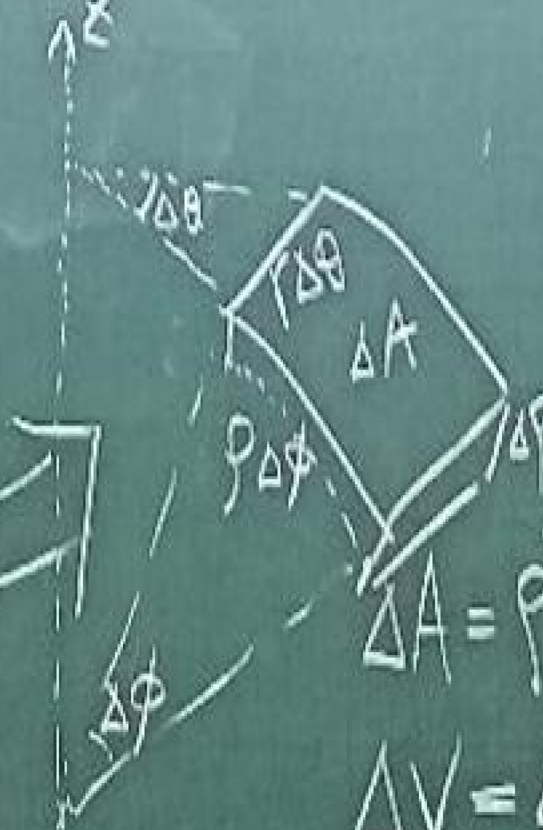
$$V=\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{r^2} dz\,d\theta\,r\,dr$$

$$=8\pi$$

II Spherical Coordinates

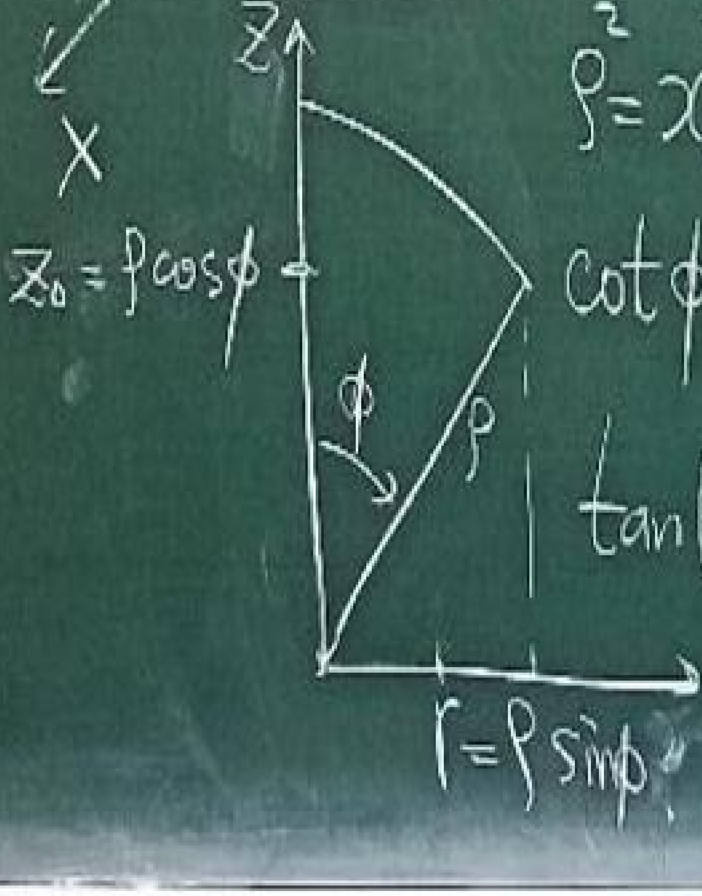
ρ, θ, ϕ





$$\Delta A = \rho^2 \sin\phi \, \Delta\theta \, \Delta\phi$$

$$\Delta V = \Delta A \, \Delta\rho = \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$$



$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot\phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan\theta = \frac{y_0}{x_0}$$

$$x_0 = r\cos\theta = \rho\sin\phi\cos\theta$$

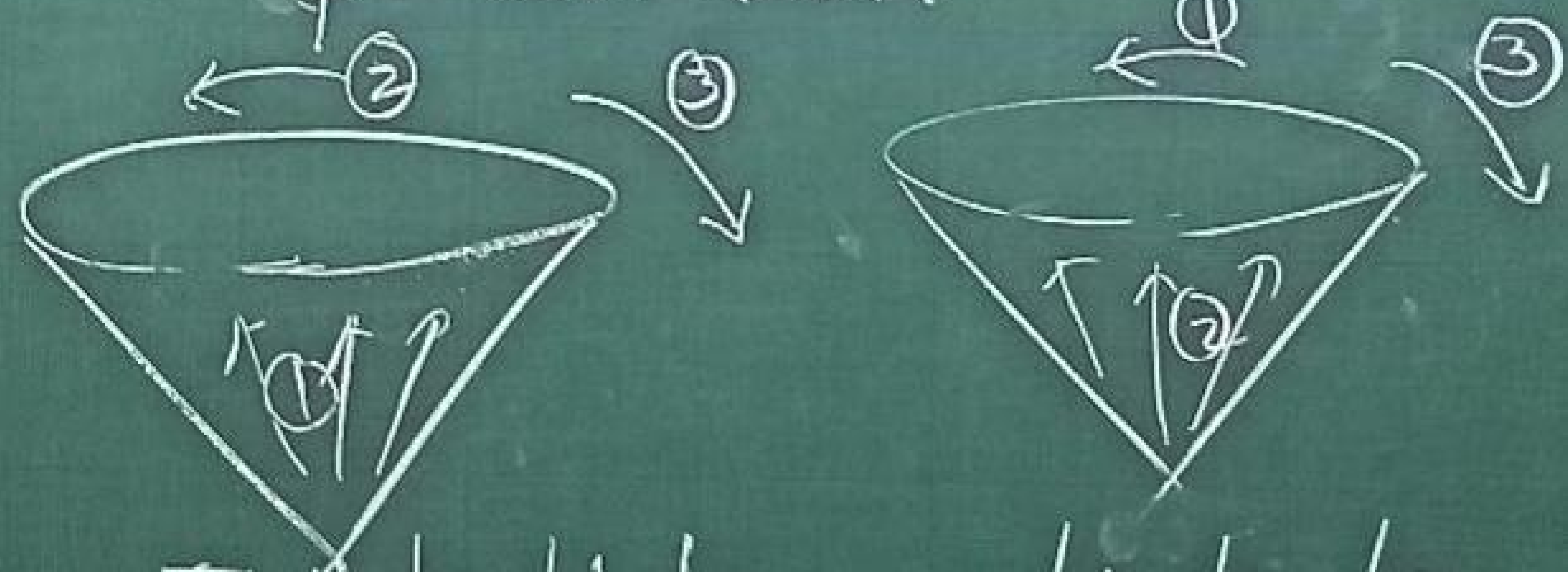
$$y_0 = r\sin\theta = \rho\sin\phi\sin\theta$$

$$z_0 = \rho\cos\phi$$

Cross Sections for various order of integration

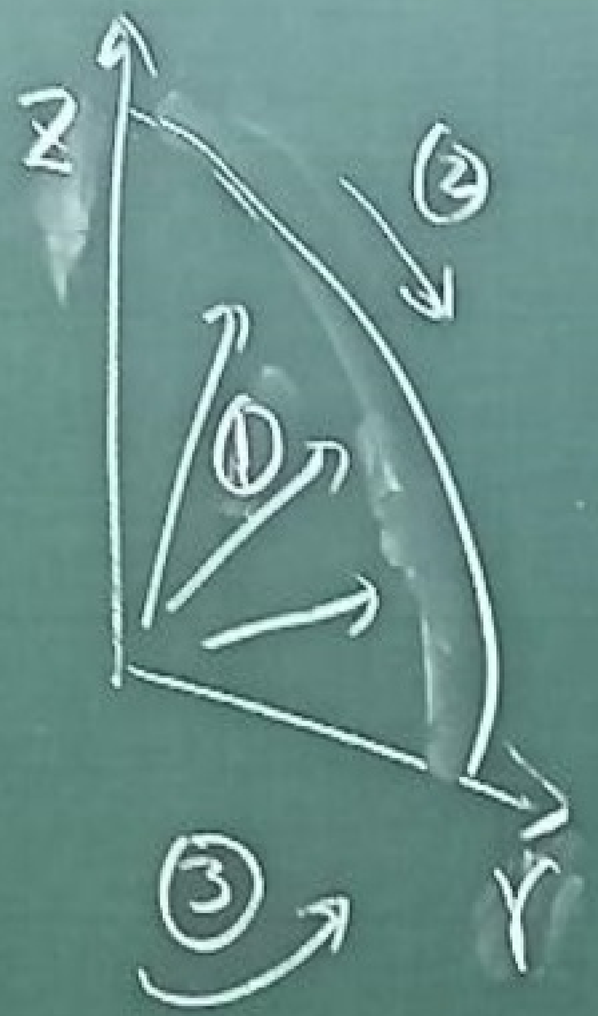
Case I: $d\rho \, d\theta \, d\phi$ or $d\theta \, d\rho \, d\phi$

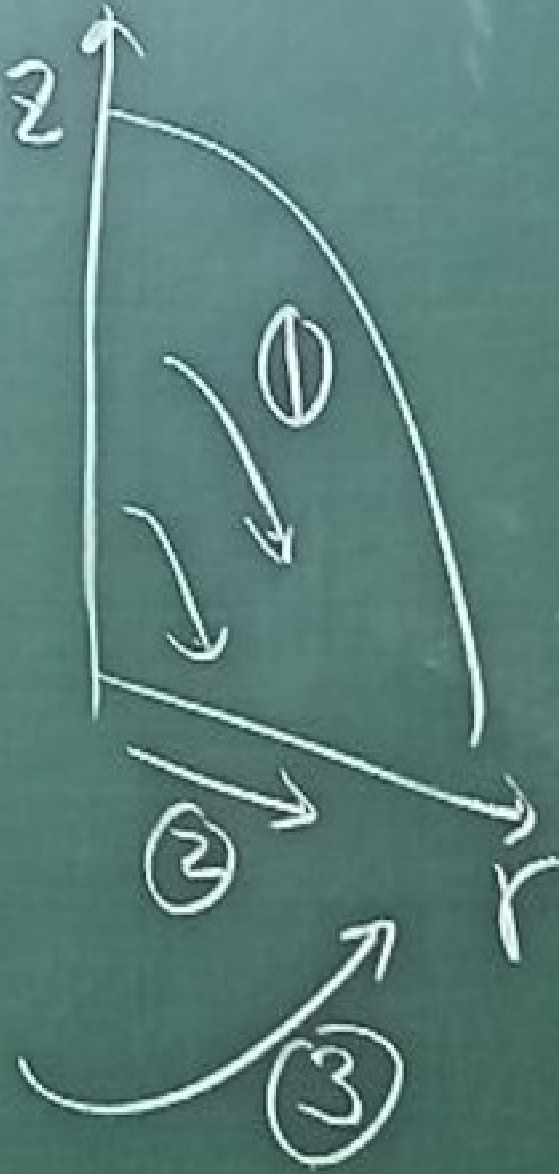
$\phi = \text{constant}$



Case II $d\theta \, d\phi \, d\rho$ $\quad$ $d\phi \, d\theta \, d\rho$

Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$

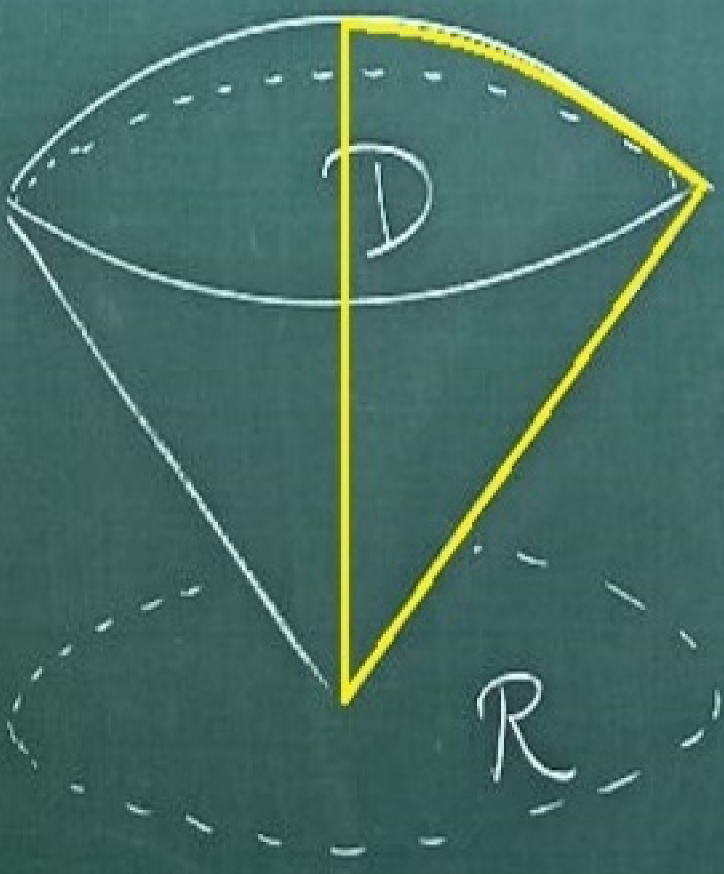




Easiest case in general
Must learn!

Example. $D=\left\{\begin{array}{l} x^2+y^2+z^2\le 1 \\ z\ge\sqrt{\dfrac{x^2+y^2}{3}}\end{array}\right\}$

Find volume of D



Sol (I). Cylindrical coordinate

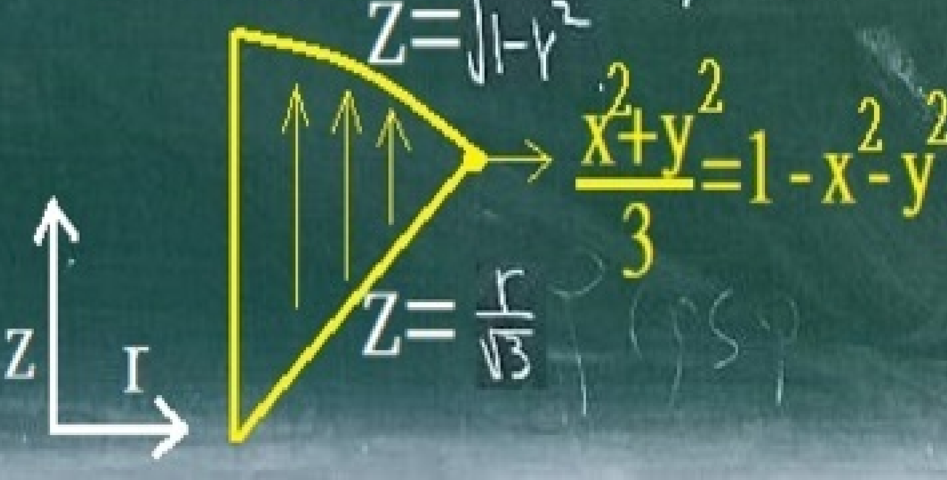
$$R=\left\{\begin{array}{l}(x,y)\ \exists\, z\in\mathbb{R}, \\ \sqrt{\dfrac{x^2+y^2}{3}}\le z\le\sqrt{1-(x^2+y^2)}\end{array}\right\}$$

$$=\left\{\frac{x^2+y^2}{3}\le 1-(x^2+y^2)\right\}$$

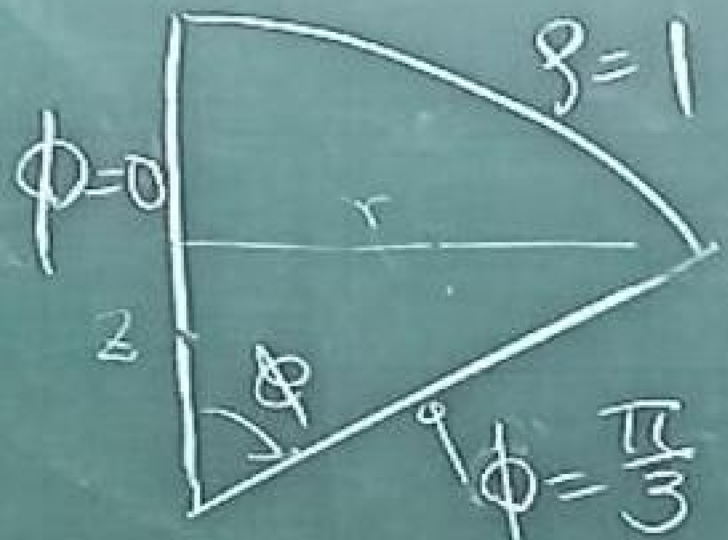
$$=\left\{x^2+y^2\le\frac{3}{4}\right\}$$

$$V=\iint_R\int_{\frac{r}{\sqrt3}}^{\sqrt{1-r^2}}dz\,dA$$

$$=\int_0^{2\pi}\int_{r=0}^{\frac{\sqrt3}{2}}\left(\sqrt{1-r^2}-\frac{r}{3}\right)r\,dr\,d\theta$$



II Spherical Coordinate



$(x^2+y^2+z^2=1)$

$$\left(\frac{r}{z}=\sqrt{3},\ \tan\phi=\sqrt{3}\right)$$

$$z=\sqrt{\frac{x^2+y^2}{3}}$$

$$D=\left\{\begin{array}{l} 0\le\rho\le 1 \\ 0\le\phi\le\frac{\pi}{3} \\ 0\le\theta\le 2\pi \end{array}\right\}$$

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\underbrace{\frac{1}{3}}_{d\rho}\underbrace{\left(1-\frac{1}{2}\right)}_{d\phi}\underbrace{2\pi}_{d\theta}=\frac{\pi}{3}$$