

Triple Integrals in Cylindrical and Spherical Coordinates

I. Cylindrical Coordinate

= polar coordinate + z coord.

$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}, \quad z = z$$

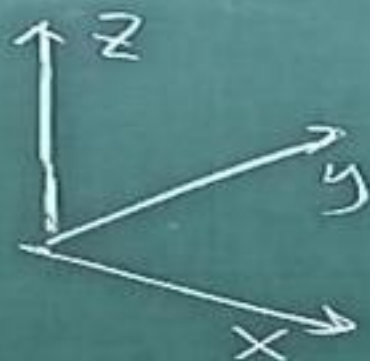
$$\theta = \begin{cases} \tan^{-1} \frac{y}{x} \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right), & (x, y) \in \text{I, IV} \\ \tan^{-1} \frac{y}{x} \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) & (x, y) \in \text{II, III} \end{cases}$$

$$dV = dA \cdot dz = r dr d\theta dz$$



Finding limits of integration

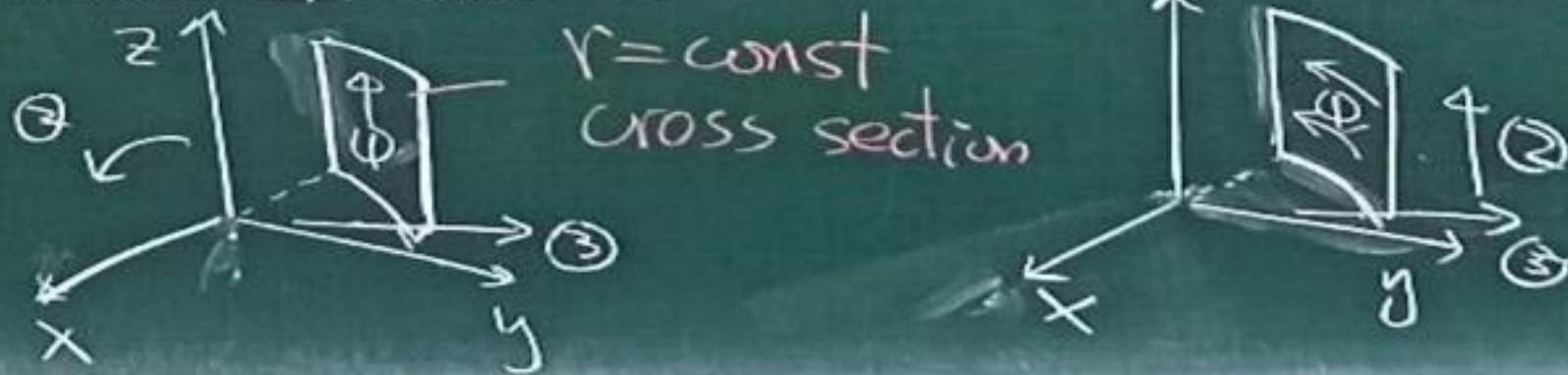
Case I: $r dr d\theta dz$ ($d\theta r dr dz$ is similar)



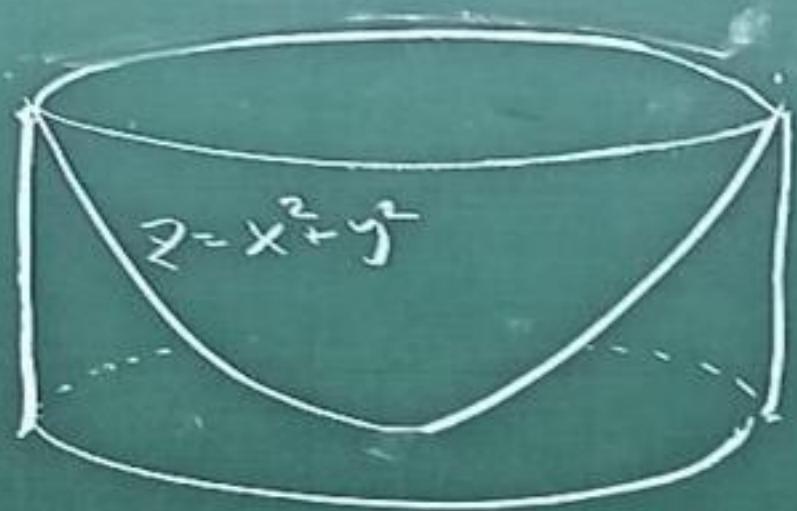
Case II $r dr dz d\theta$ or $dz r dr d\theta$



Case III $dz d\theta r dr$ or $d\theta dz r dr$



Eg $D = \left\{ \begin{array}{l} x^2 + y^2 \leq 4 \\ 0 \leq z \leq x^2 + y^2 \end{array} \right\}$



Find volume of D using cylindrical coordinates.

case I $r dr d\theta dz$

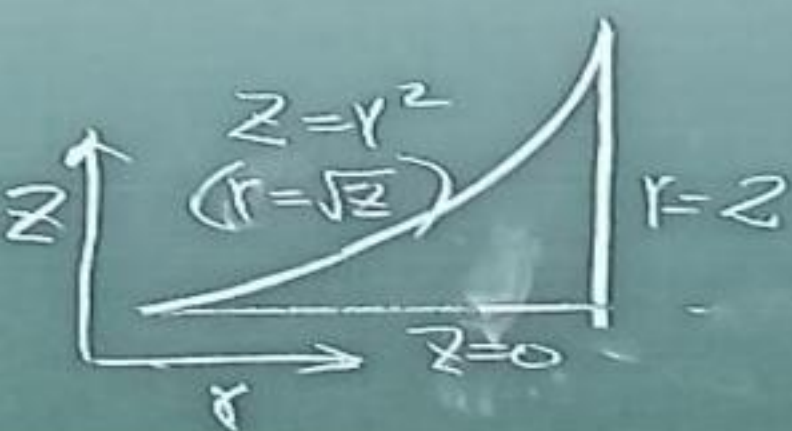
Step 1 cross section



$$V = \int_0^4 \int_0^{2\pi} \int_{r=\sqrt{z}}^2 r dr d\theta dz$$

Case II

$r dr dz d\theta$ or $dz r dr d\theta$



$$\begin{aligned}
 V &= \int_0^{2\pi} \int_0^4 \int_{y^2}^2 r dr dz d\theta \\
 &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{z}} dz r dr d\theta \\
 &= 8\pi
 \end{aligned}$$

Case III

$dz d\theta r dr$



$$\begin{aligned}
 V &= \int_{r=0}^2 \int_{\theta=0}^{2\pi} \int_{z=0}^{r^2} dz d\theta r dr \\
 &= 8\pi
 \end{aligned}$$