# Triple Integrals in Cylindrical and Spherical Coordinates

## I. Cylindrical Coordinate

= polar coordinate + $z$ coord.

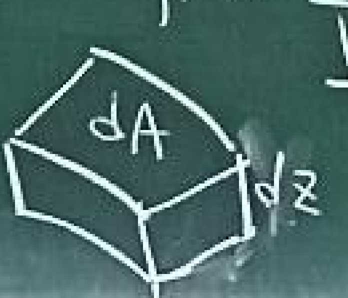
$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

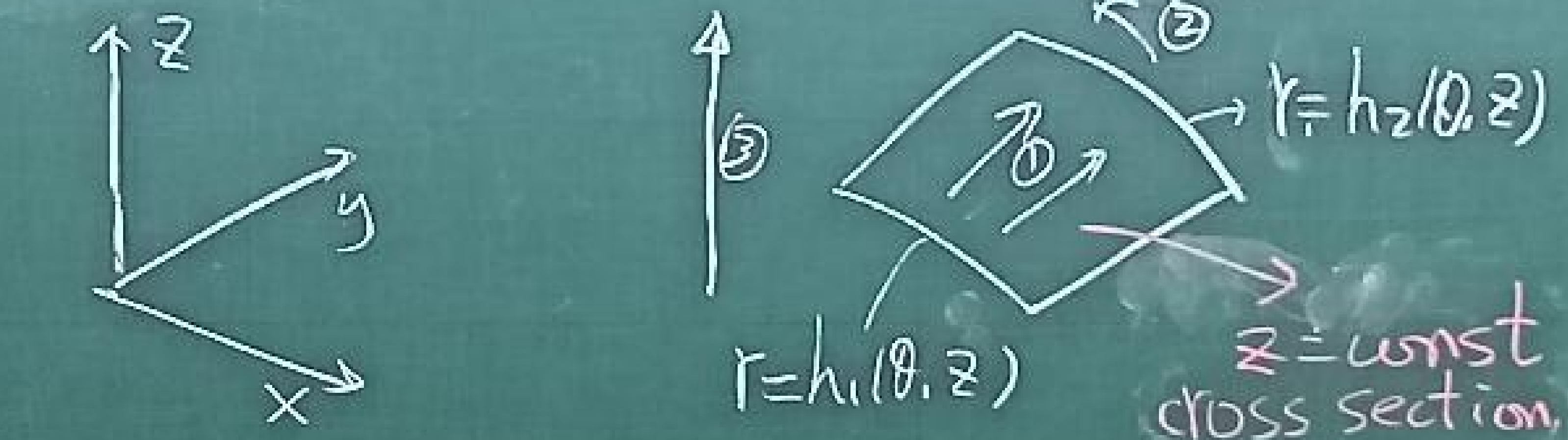
$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \mathrm{I}, \mathrm{IV} \\ \tan^{-1}\frac{y}{x} \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \qquad\qquad\quad \left(\frac{-3\pi}{2}, \frac{\pi}{2}\right) & (x,y) \in \frac{\mathrm{II}}{\mathrm{III}} \end{cases}$$

$$dV = dA \cdot dz = r\,dr\,d\theta\,dz$$

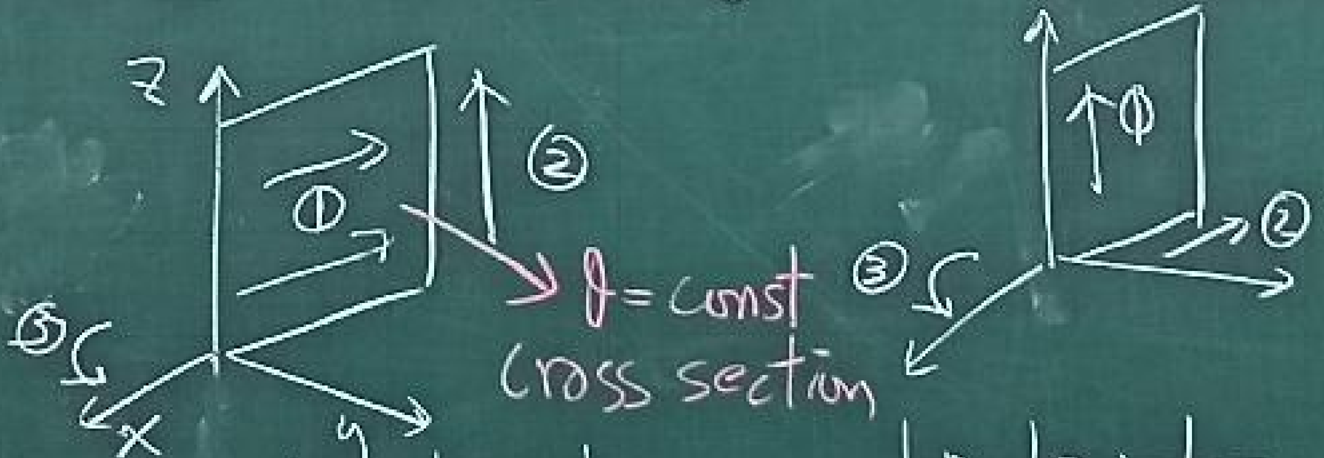


# Finding limits of integration

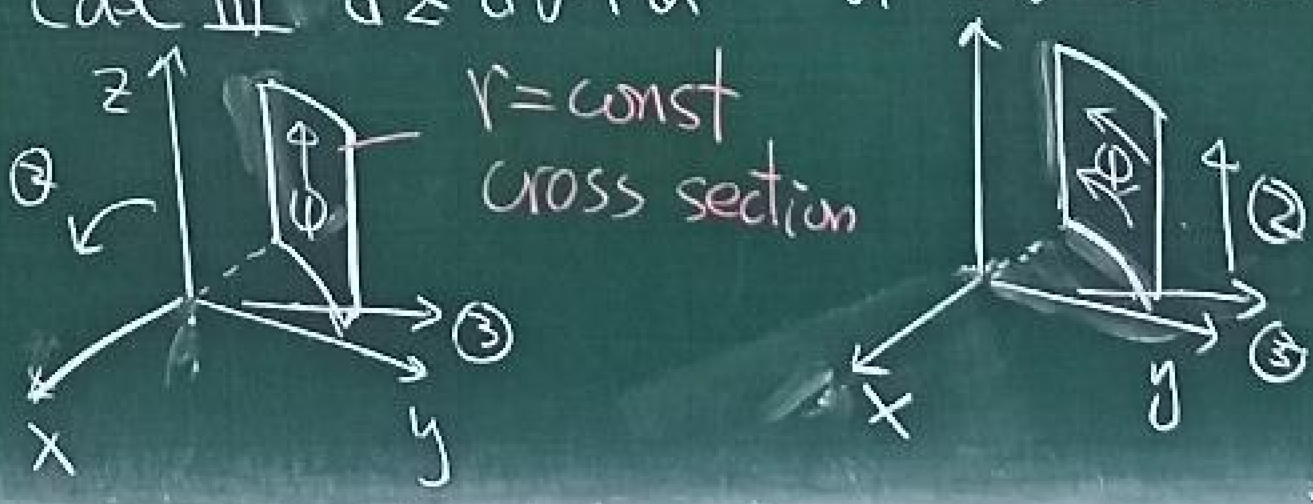
Case I: $r\,dr\,d\theta\,dz$ ($d\theta\, r\,dr\,dz$ is similar)



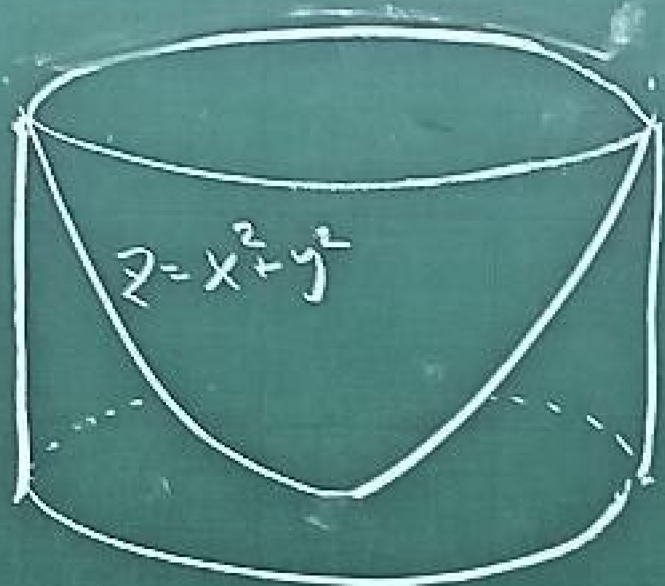

Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$




Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$



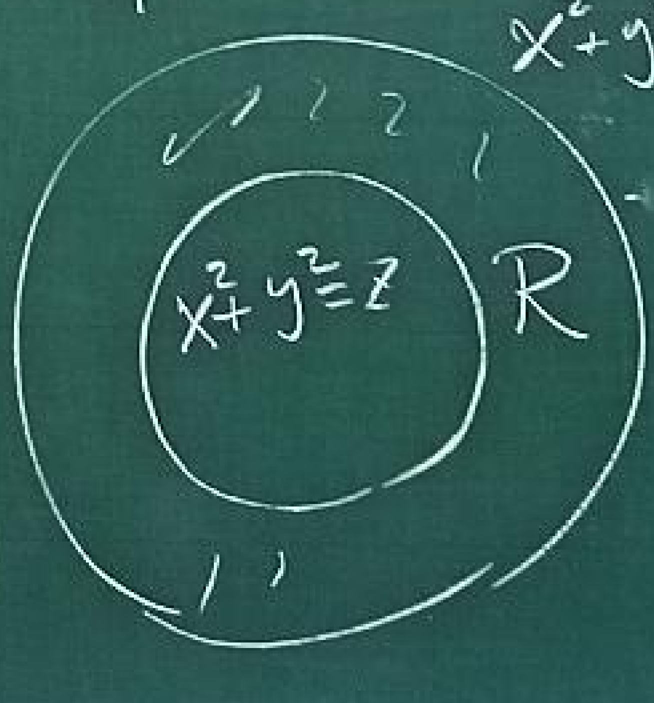
Eg $D = \left\{ \begin{matrix} x^2+y^2 \le 4 \\ 0 \le z \le x^2+y^2 \end{matrix} \right\}$




Find volume of D using Cylindrical coordinates.

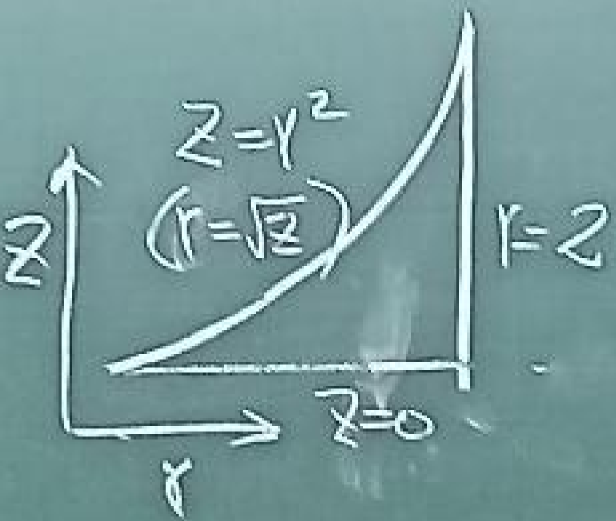
case I rdrdθdz

Step 1 cross section




$$V = \int_0^4 \int_0^{2\pi} \int_{r=\sqrt{z}}^{2} r\,dr\,d\theta\,dz$$

Case $\underline{\text{II}}$ $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

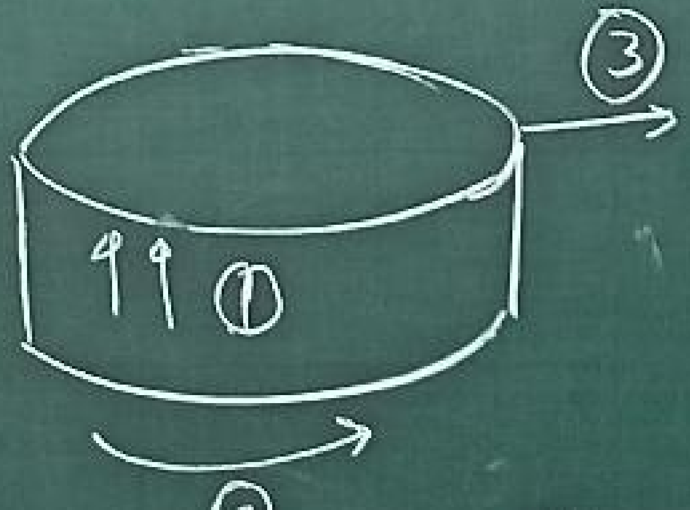



$$V=\int_0^{2\pi}\int_0^{4}\int_{r=\sqrt{z}}^{2} r\,dr\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2}\int_{z=0}^{r^2} dz\,r\,dr\,d\theta$$

$$=8\pi$$

Case $\underline{\text{III}}$ $dz\,d\theta\,r\,dr$




$$V=\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{r^2} dz\,d\theta\,r\,dr$$

$$=8\pi$$