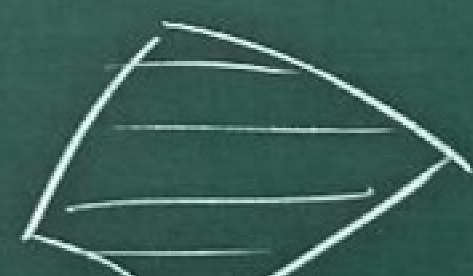
Integration in polar coordinate.

Eg 1 $\iint_R f(x,y)\,dA$

R = region enclosed by $\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2+y^2=1 \\ x^2+y^2=4 \end{cases}$

Both  $(dx\,dy)$

and  $(dy\,dx)$

are complicated.

In this example

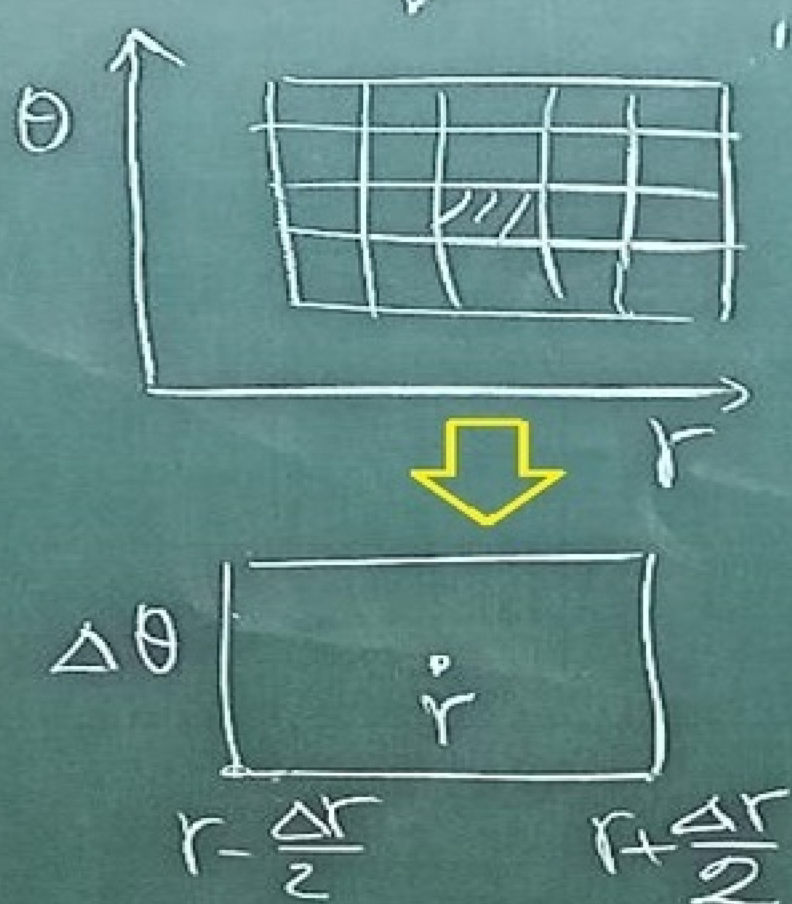
$$R = \left\{ 1 \leq r \leq 2,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \right\}$$

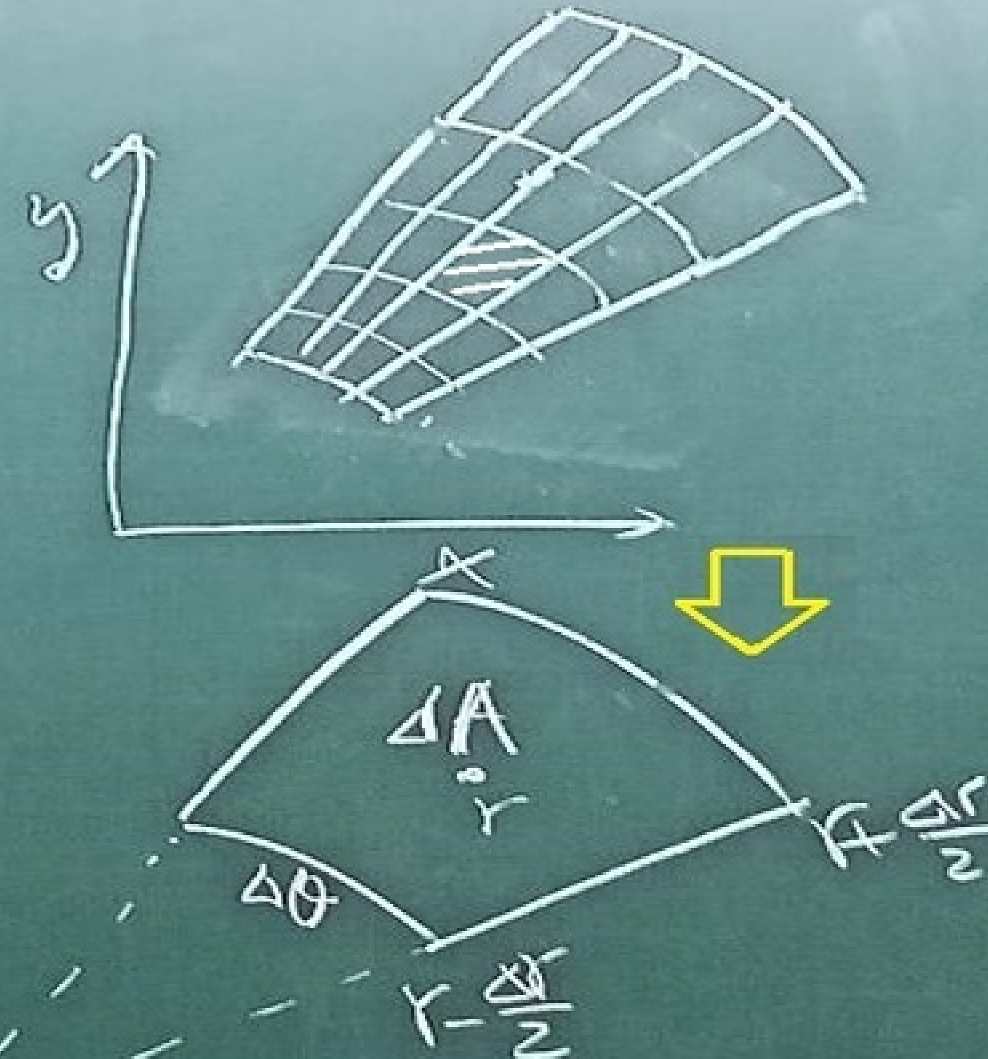
$$I = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} f(\underset{\overset{\shortparallel}{x}}{r\cos\theta}, \underset{\overset{\shortparallel}{y}}{r\sin\theta})\, dA$$

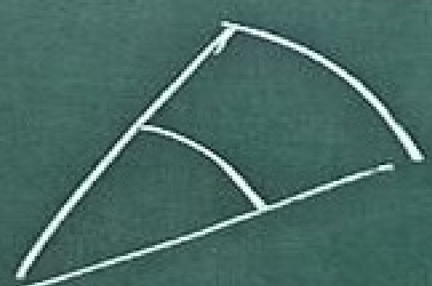
$$dA \stackrel{?}{=} dr\, d\theta$$

<u>Ans</u>: $dA = r\, dr\, d\theta$

Why?





$\Delta A =$  $-$ 

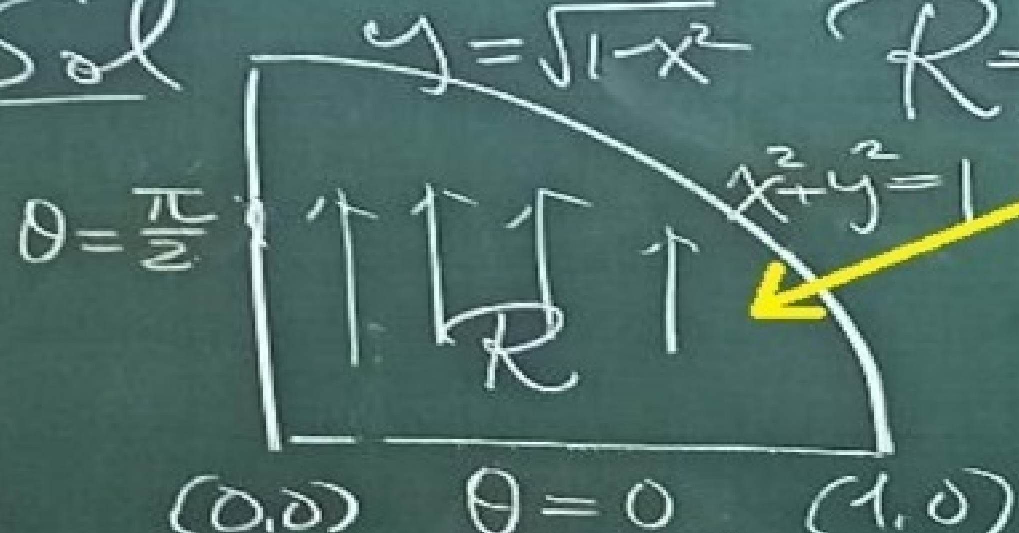
$$= \pi\left(r+\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi} - \pi\left(r-\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi}$$

$$= r\,\Delta r\,\Delta\theta$$

Integration in Polar Coordinates

Eg 2 $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

Sol

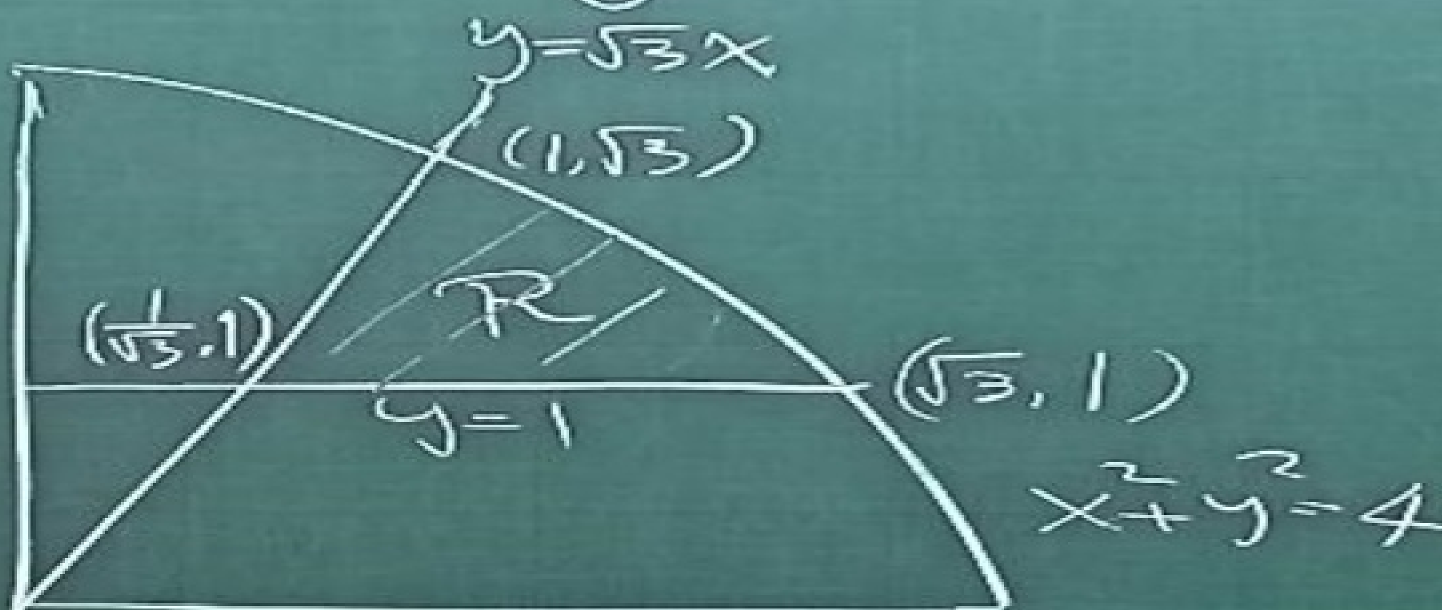


$$R=\left\{\begin{matrix} 0\le y\le\sqrt{1-x^2} \\ 0\le x\le 1\end{matrix}\right\} = \left\{\begin{matrix} 0\le r\le 1 \\ 0\le\theta\le\frac{\pi}{2}\end{matrix}\right\}$$

$$I=\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\,(\underbrace{r\,dr\,d\theta}_{= dA})$$

$$=\left(\int_0^1 r^3\,dr\right)\cdot\left(\int_0^{\frac{\pi}{2}} d\theta\right)=\frac{\pi}{8}$$

Eg 3 Find the area enclosed by $\begin{cases} y=\sqrt{3}x \\ y=1 \\ x^2+y^2=4 \end{cases}$ in 1st quadrant

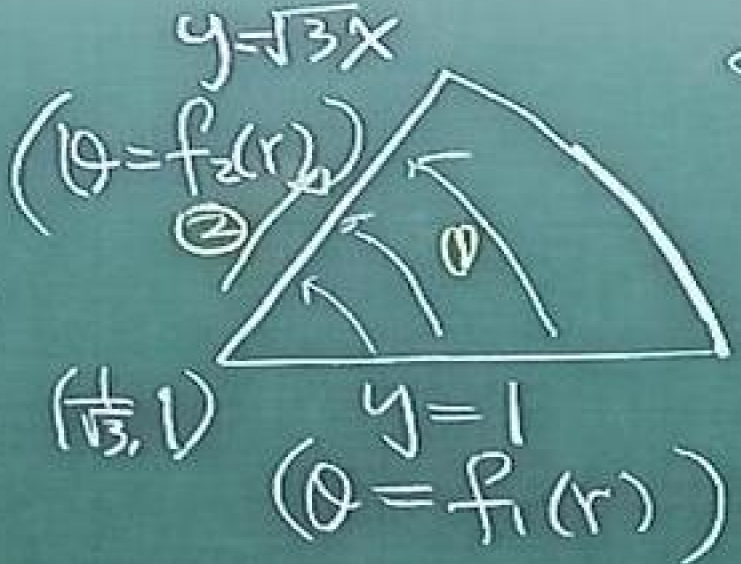


Sol: Method 1:

A =  − 

$$= \pi 2^2 \cdot \frac{\frac{\pi}{6}}{2\pi} - \frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\cdot 1$$

Method 2: Polar Cardinate



$$R = \left\{ \begin{array}{l} f_1(r) \leq \theta \leq f_2(r) \\ \frac{2}{\sqrt{3}} \leq r \leq 2 \end{array} \right\}$$

$$\frac{2}{\sqrt{3}} = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1^2}$$

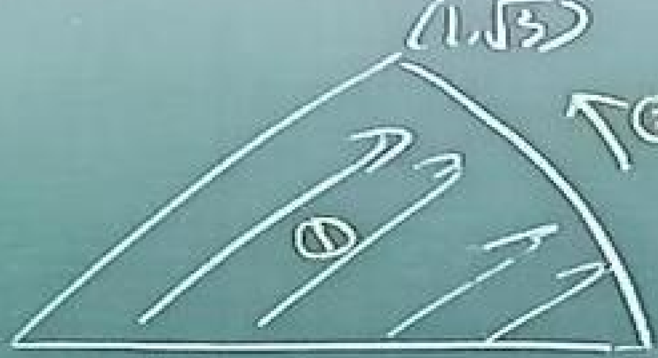
$$y = 1 \Longleftrightarrow r\sin\theta = 1 \Rightarrow \theta = \sin^{-1}\left(\frac{1}{r}\right) = f_1(r)$$

$$y = \sqrt{3}x \Longleftrightarrow \theta = \frac{\pi}{3} = f_2(r)$$

$$I = \int_{r=\frac{2}{\sqrt{3}}}^{2} \int_{\theta=\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} d\theta \, r \, dr \rightarrow \text{difficult}$$

$$= \int_{\frac{2}{\sqrt{3}}}^{2} r\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{r}\right)\right) dr = ?$$

Method 3



$$r \; ② \quad R = \left\{ \begin{array}{l} g(\theta) \le r \le 2 \\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3} \end{array} \right\}$$

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \frac{\pi}{3} = \tan^{-1}(\sqrt{3})$$

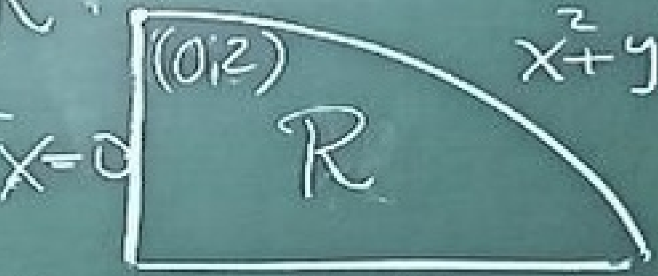
$$y = 1 \Leftrightarrow r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta} = g(\theta)$$

$$I = \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{r=\frac{1}{\sin\theta}}^{2} r \, dr \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(2 - \frac{1}{2\sin^2\theta}\right) d\theta$$

$$= \left(2\theta + \frac{\cot\theta}{2}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{1}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

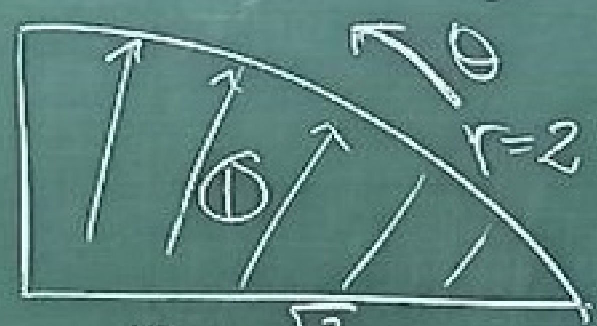
Eg 4: Evaluate $\iint_R f(x,y)\,dA$ in polar coordinate where



$x^2+y^2=4$ $(r=2)$

$(r=\sqrt{2}) \leftrightarrow (0,\sqrt{2})$ $\quad y=\sqrt{2}$ $\quad (\sqrt{2},\sqrt{2})$ $(\theta=\frac{\pi}{4})$

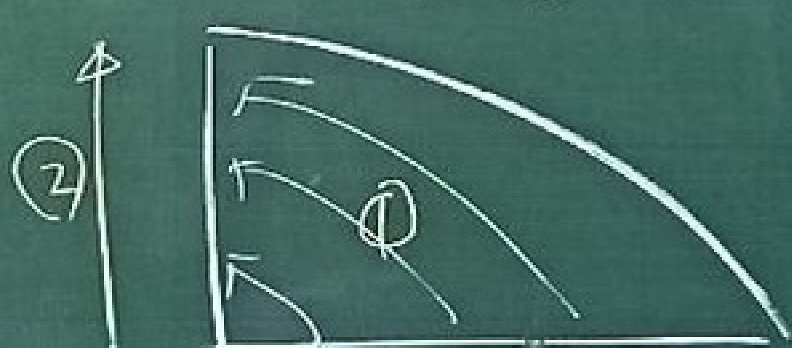
Sol: Method ⓐ $r\,dr\,d\theta$



$r=\dfrac{\sqrt{2}}{\sin\theta}$

$$I=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=\frac{2}{\sin\theta}}^{2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Method ⓑ $d\theta\, r\,dr$



$\theta=\sin^{-1}\left(\dfrac{\sqrt{2}}{r}\right)$

$$I=\int_{r=\sqrt{2}}^{2}\int_{\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)}^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,d\theta\, r\,dr$$

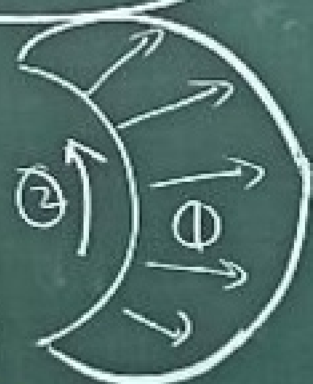
Eg 5 Evaluate $\iint_R g(r,\theta)\, dA$

where R is the region $\begin{cases} \text{inside } r = 1+\cos\theta \\ \text{outside } r = 1 \end{cases}$

$r = 1+\cos\theta \Rightarrow \cos\theta = r-1$

$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0, \frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [\frac{-\pi}{2}, 0] \end{cases}$

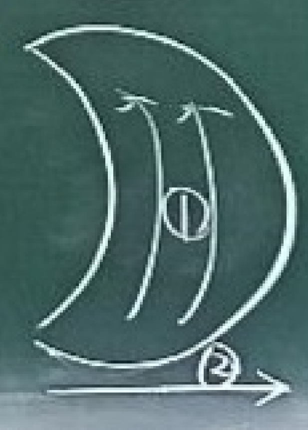
(a) $r\,dr\,d\theta$: (usually preferred)



$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

(b) $d\theta\, r\,dr$

(usually not good)



$$I = \int_{r=1}^{2} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r,\theta)\, d\theta\, r\,dr$$

Remark

$r\,dr\,d\theta$ is usually a better choice than $d\theta\, r\,dr$ since most curves in polar coordinates appear naturally in the form $r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r\,dr$

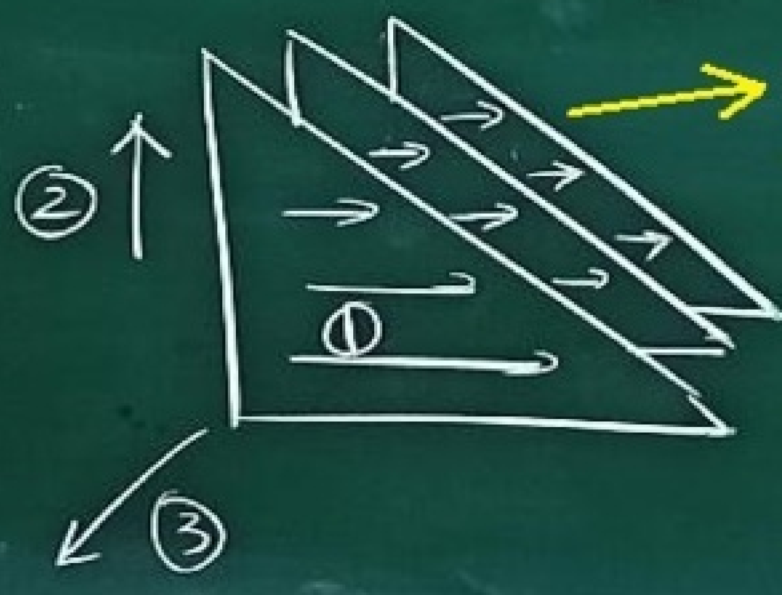
Triple integrals in Cartesian Coordinates

$$I = \iiint_D f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_{k=0}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

When $f(x,y,z) \equiv 1$, $I = \text{volume of } D$

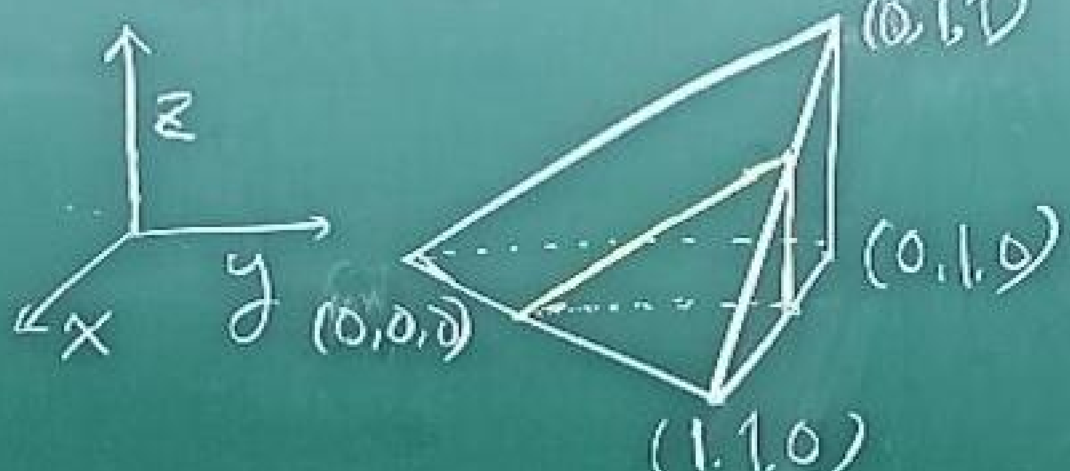
In Cartesian Coordinates

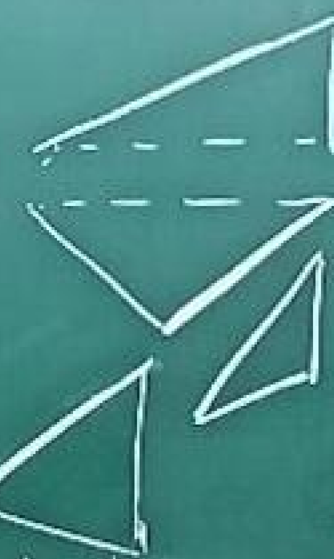
$$dV = dx\,dy\,dz = dy\,dz\,dx = \cdots\cdots$$



③ = constant

Eg1 Find the volume of the domain enclosed by





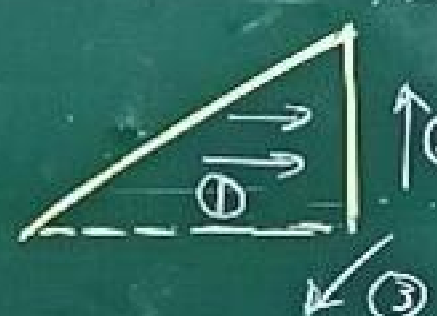
$x=0$ (A)

$z=0$ (B)

$y=1$ (C)

$x-y+z=0$ (D)

Method 1: $\overset{①}{dy}\,\overset{②}{dz}\,\overset{③}{dx}$ (cut along ③: $x=$constant)



$$I=\int_{x=0}^{1}\int_{z=0\,(B)}^{\substack{z=1-x\\(C\cap D)}}\int_{y=x+z\,(D)}^{y=1\,(C)} 1\,dy\,dz\,dx$$

① dy: need (C) and (D) in the form $y=f(x,z)$
$y=1$ $y=x+z$

② dz: need (B) and $(C\cap D)$ in the form $z=g(x)$
$z=0$ $z=1-x$ (eliminate y from (C),(D))

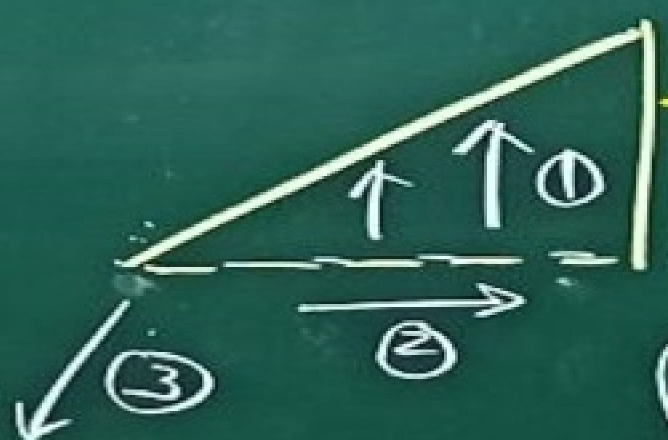
$$\therefore\ I=\int_{x=0}^{1}\int_{z=0}^{1-x}\int_{y=x-z}^{1} dy\,dz\,dx$$

$$=\int_{x=0}^{1}\int_{z=0}^{1-x}(1-x+z)\,dz\;dx$$

$$=\int_{x=0}^{1}(1-x)z+\frac{z^2}{2}\Bigg|_{z=0}^{1-x} dx$$

$$=\int_{x=0}^{1}\frac{1}{2}(1-x)^2\,dx=\frac{1}{6}$$

Method 2 : $\underset{①}{dz}\,\underset{②}{dy}\,\underset{③}{dx}$



Cut along ③=const

Step ①:

$$\int_{(B)}^{(D)} 1\,dz=\int_{z=0}^{z=y-x} dz$$
$$=y-x$$

Need (B).(D) as $z=F(x,y)$

Step ②

$$\int_{B\cap D}^{C} (y-x)\,dy = \int_{y=x}^{1} (y-x)\,dy$$

(Need $B\cap D$ and C as $y=G(x)$
by eliminating z ($\Leftarrow$ ①))

$$= \left[\frac{y^2}{2} - xy\right]_{y=x}^{1} dy = \frac{(1-x)^2}{2}$$

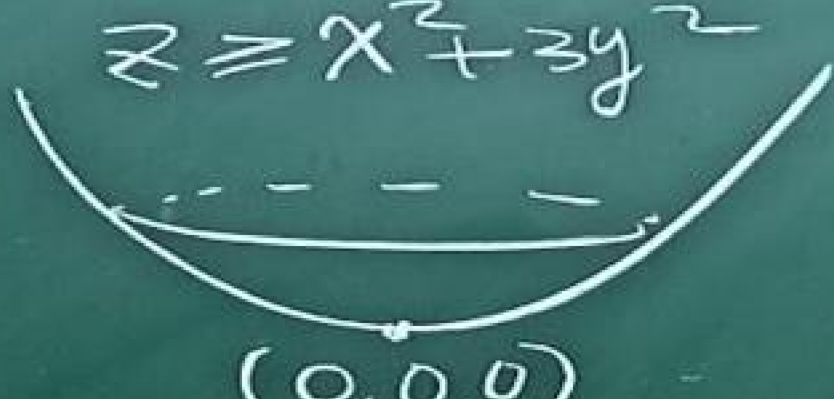
Step ③ $\displaystyle\int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{6}$

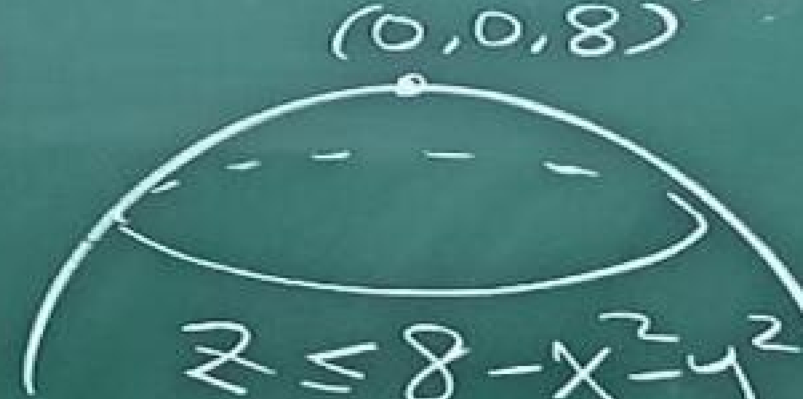
Other orders: $dx\,dy\,dz$,
$dy\,dz\,dx$, . . .

(Homework)

Eg2. Find the volume of

$$D=\{x^2+3y^2 \le z \le 8-x^2-y^2\}$$



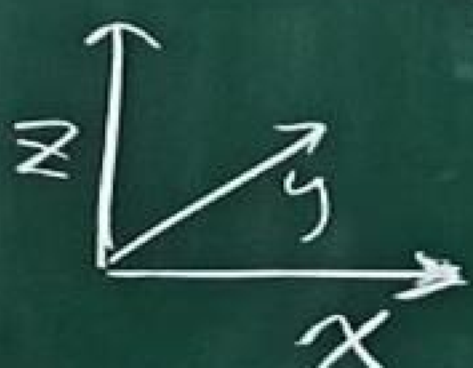


③ = ?

Not z!

$$\begin{cases} d\underset{z}{①}\, d\underset{y}{②}\, dx \\ \text{or} \\ \text{or } d\underset{z}{①}\, d\underset{x}{②}\, dy \end{cases}$$

are preferred (easier)



For example $dz\,dy\,dx$

$$V = \iint_R \int_{z=x^2+3y^2}^{8-x^2-y^2} 1 \, dz \, dA$$

$$R = \{(x,y) \mid x^2+3y^2 \le 8-x^2-y^2\}$$

(So that $\int_{z=x^2+y^2}^{8-x^2-y^2} dz$ is meaningful (positive))

$$= \{x^2+2y^2 \le 4\}$$

$$V = \iint_{x^2+2y^2 \le 4} 8-2x^2-4y^2 \, dy \, dx$$

(Rm. Let $Y = 2y$

$dY = 2dy$

$$\iint_{x^2+Y^2 \le 4} 8-2(x^2+Y^2)\frac{dY}{2}\,dx$$

Polar coordinate integration)

$$\iint dy\,dx = \int_{x=?}^{?}\int_{y=?}^{?} dy\,dx$$

$$x^2+2y^2 \leq 4$$

$$\Rightarrow -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}$$

$$R = \left\{ \begin{array}{l} -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}} \\ -2 \leq x \leq 2 \end{array} \right\}$$

$$\therefore V = \int_{-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-y^2)\,dy\,dx$$

$$= \int_{-2}^{2} (8-2x^2)y - \frac{4y^3}{3} \Bigg|_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \frac{8}{3}(4-x^2)\sqrt{\frac{4-x^2}{2}}\,dx \quad \left(\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \end{array} \right)$$