Integration in polar coordinate.

Eg 1 $\displaystyle\iint_R f(x,y)\,dA$

R = region enclosed by $\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2+y^2 = 1 \\ x^2+y^2 = 4 \end{cases}$

Both

(dxdy)

and

(dy dx)

are complicated.

In this example

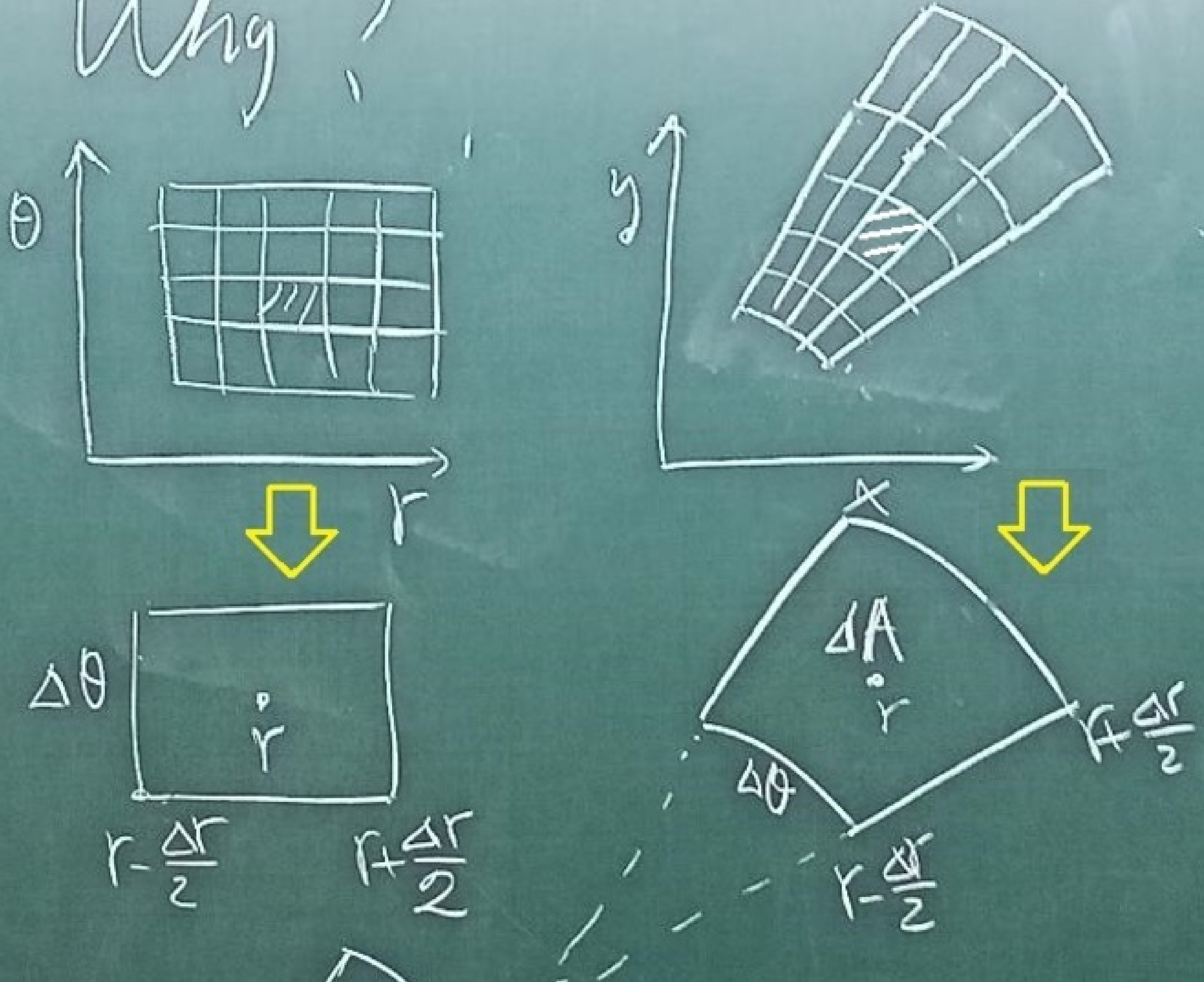
$$R=\left\{1\le r\le 2,\ \frac{\pi}{4}\le\theta\le\frac{\pi}{3}\right\}$$

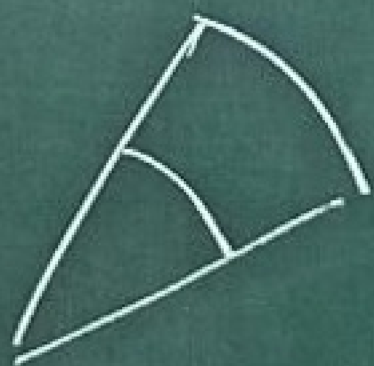
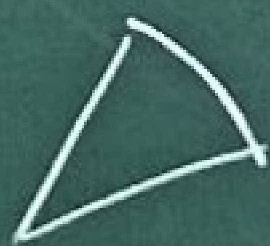
$$I=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} f(\underset{\substack{\shortparallel\\ x}}{r\cos\theta},\ \underset{\substack{\shortparallel\\ y}}{r\sin\theta})\,dA$$

$$dA \stackrel{?}{=} dr\,d\theta$$

<u>Ans</u>: $dA=r\,dr\,d\theta$

Why?



$\Delta A =$  $-$ 

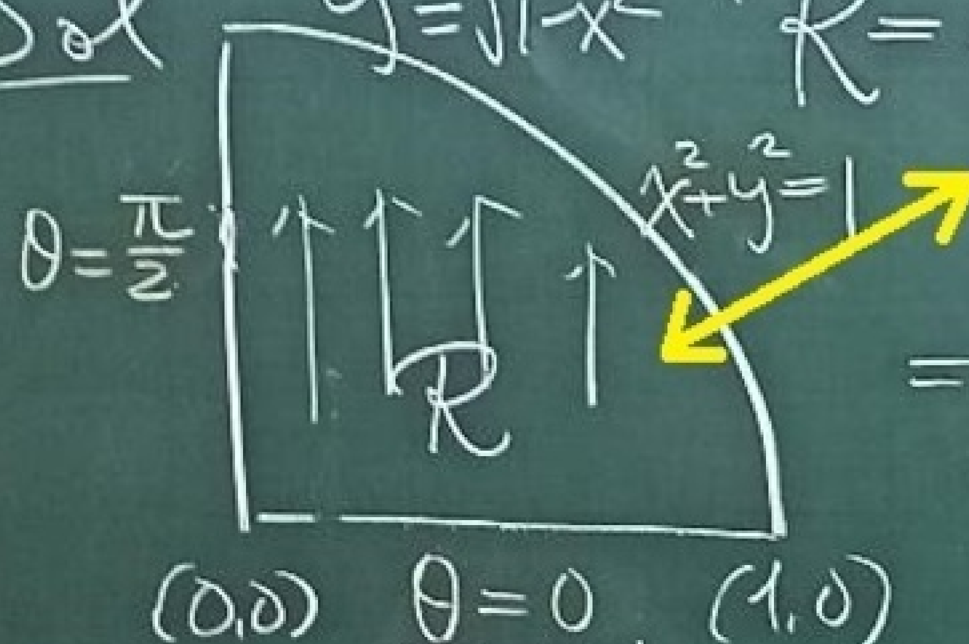
$$= \pi\left(r+\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi} - \pi\left(r-\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi}$$

$$= r\,\Delta r\,\Delta\theta$$

Integration in Polar Coordinates

<u>Eg</u> 2 $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx$

<u>Sol</u>



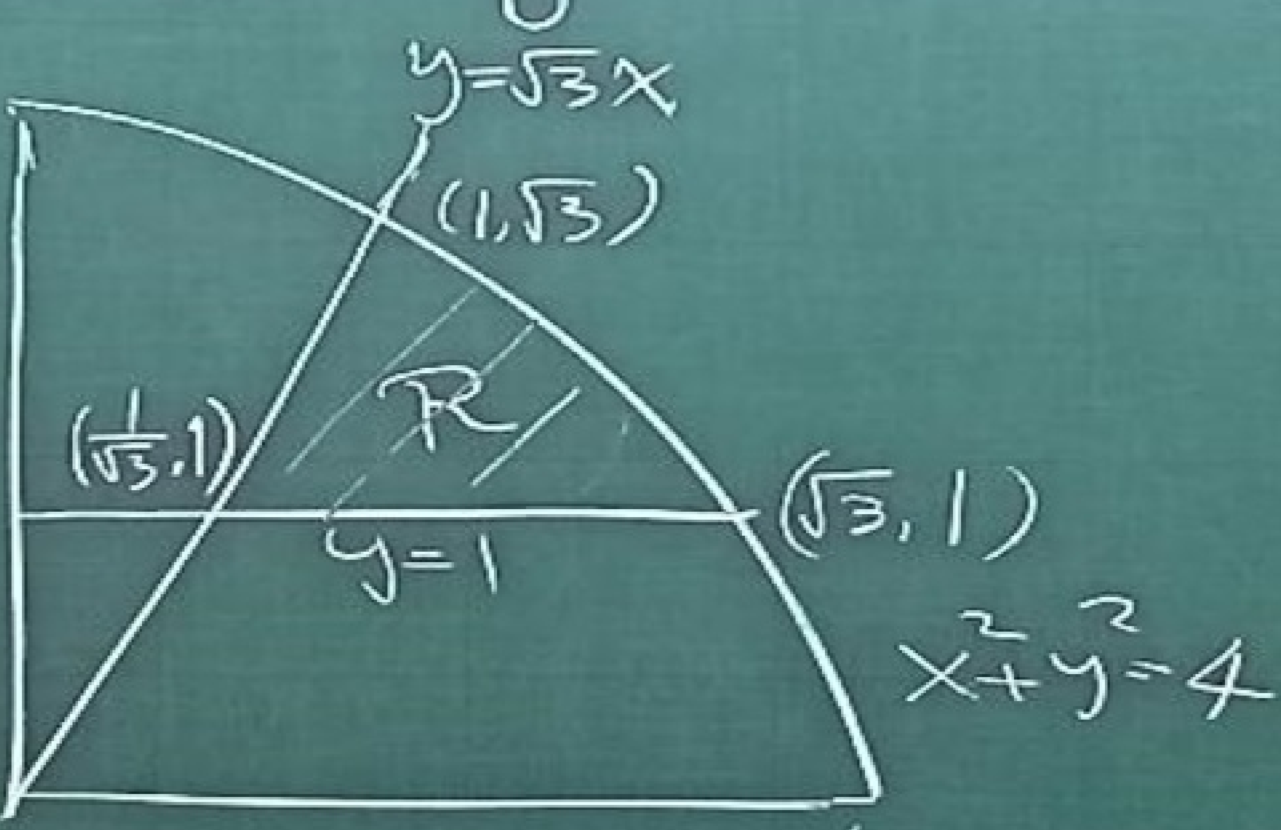
$$R=\left\{\begin{matrix}0\le y\le\sqrt{1-x^2}\\ 0\le x\le 1\end{matrix}\right\}$$

$$=\left\{\begin{matrix}0\le r\le 1\\ 0\le \theta\le \frac{\pi}{2}\end{matrix}\right\}$$

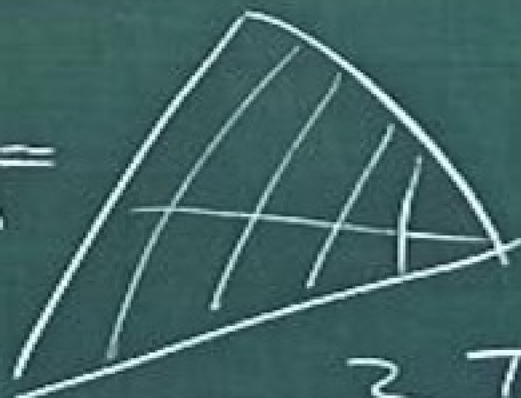
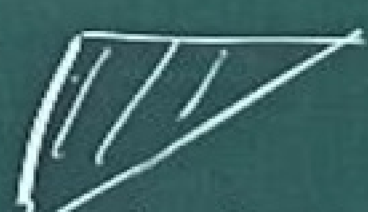
$$I=\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\,\underset{dA}{\underset{\shortparallel}{(r\,dr\,d\theta)}}$$

$$=\left(\int_0^1 r^3\,dr\right)\cdot\left(\int_0^{\frac{\pi}{2}}d\theta\right)=\frac{\pi}{8}$$

Eg 3 Find the area enclosed by $\begin{cases} y=\sqrt{3}x \\ y=1 \\ x^2+y^2=4 \end{cases}$ in 1st quadrant

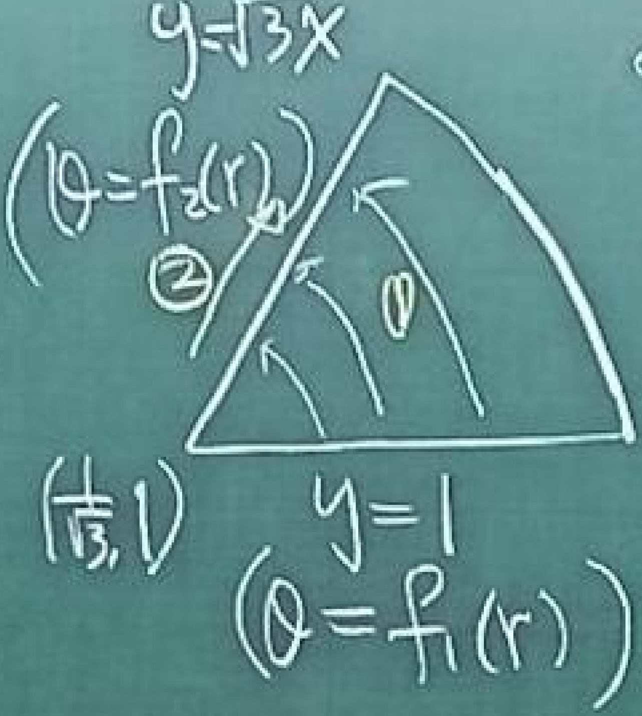


Sol: Method 1:

A =  − 

$$= \pi 2^2 \cdot \frac{\frac{\pi}{6}}{2\pi} - \frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\cdot 1$$

Method 2: Polar Coordinate



$$R=\left\{\begin{array}{l} f_1(r)\le\theta\le f_2(r) \\ \underset{\displaystyle \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1^2}}{\underset{\shortparallel}{\frac{2}{\sqrt{3}}}}\le r\le 2 \end{array}\right\}$$

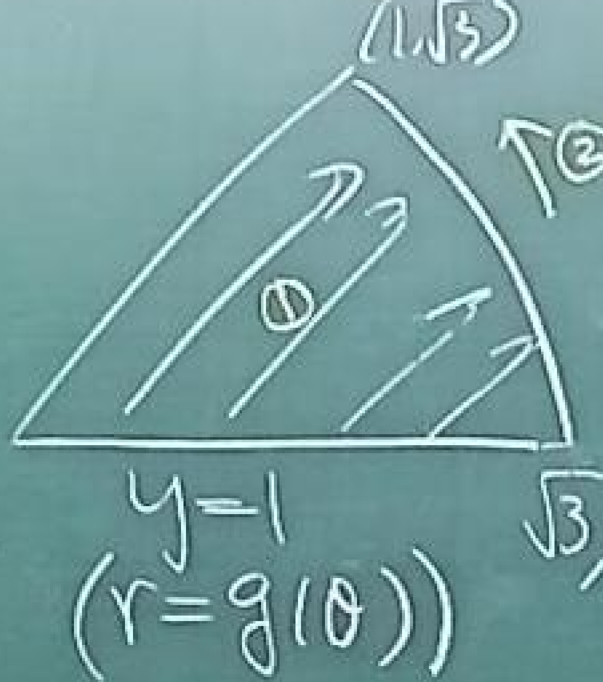
$$y=1 \Leftrightarrow r\sin\theta=1 \Rightarrow \theta=\sin^{-1}\left(\frac{1}{r}\right)$$
$$=f_1(r)$$

$$y=\sqrt{3}x \Leftrightarrow \theta=\frac{\pi}{3}=f_2(r)$$

$$I=\int_{r=\frac{2}{\sqrt{3}}}^{2}\int_{\theta=\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} d\theta\, r\, dr \rightarrow \text{difficult}$$

$$=\int_{\frac{2}{\sqrt{3}}}^{2} r\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{1}{r}\right)\right)dr=?$$

Method 3



$$R=\left\{\begin{array}{l} g(\theta)\le r\le 2 \\[2ex] \underset{\substack{\| \\ \tan^{-1}\left(\frac{1}{\sqrt3}\right)}}{\frac{\pi}{6}} \le \theta \le \underset{\substack{\| \\ \tan^{-1}(\sqrt3)}}{\frac{\pi}{3}} \end{array}\right\}$$

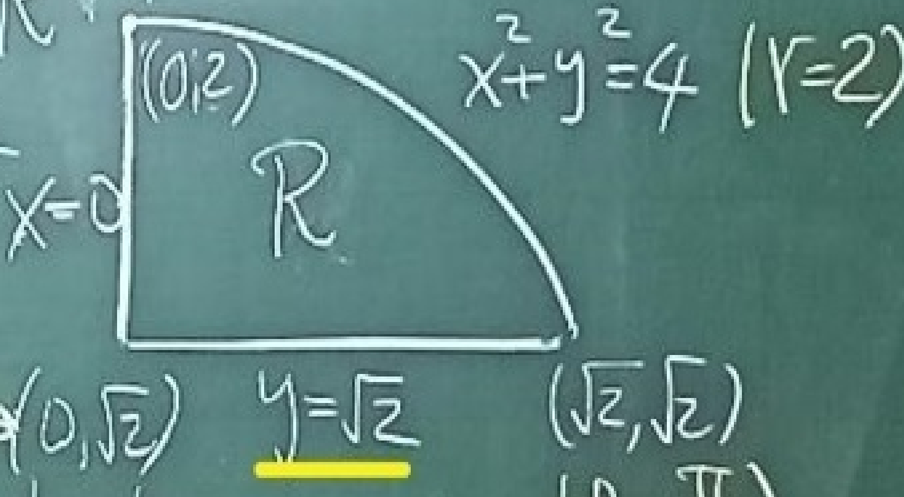
$$y=1 \Leftrightarrow r\sin\theta=1 \Rightarrow r=\frac{1}{\sin\theta}=g(\theta)$$

$$I=\int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{r=\frac{1}{\sin\theta}}^{2} r\,dr\,d\theta$$

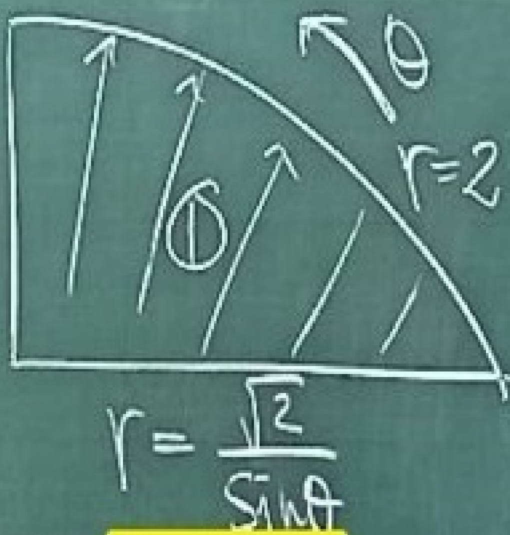
$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(2-\frac{1}{2\sin^2\theta}\right)d\theta$$

$$=\left(2\theta+\frac{\cot\theta}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{3}+\frac{1}{2}\left(\frac{1}{\sqrt3}-\sqrt3\right)$$

Eg 4: Evaluate $\iint_R f(x,y)\,dA$ in polar coordinate where

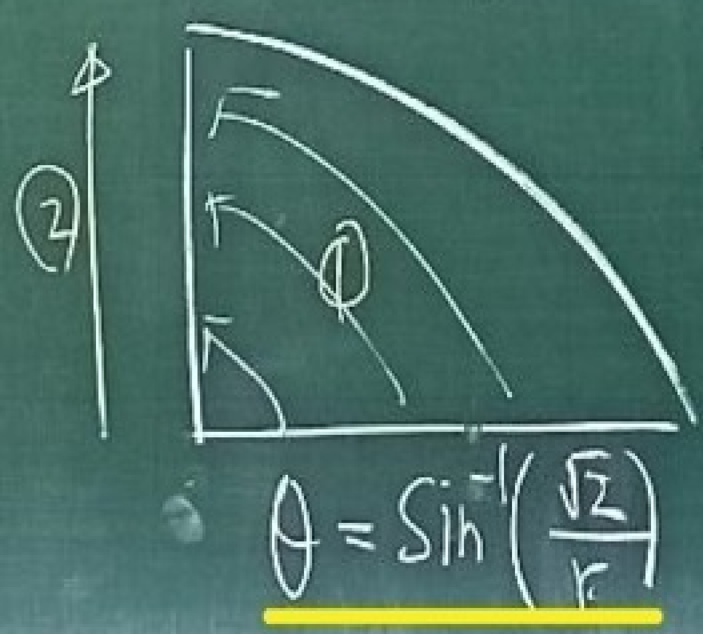


Sol: Method ⓐ $r\,dr\,d\theta$



$$I=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=\frac{2}{\sin\theta}}^{2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

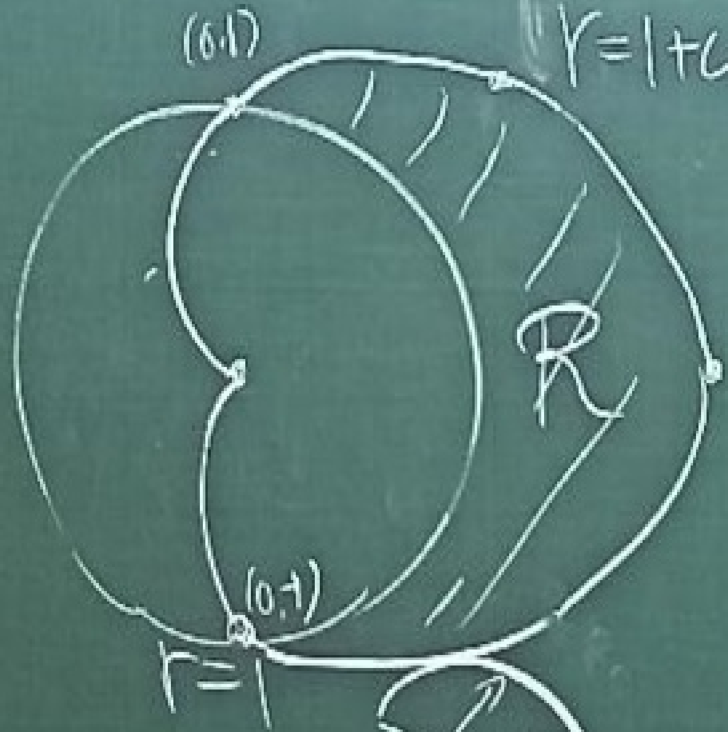
Method ⓑ $d\theta\,r\,dr$



$$I=\int_{r=\sqrt{2}}^{2}\int_{\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)}^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,d\theta\,r\,dr$$

Eg 5 Evaluate $\iint_R g(r,\theta)\,dA$

where R is the region $\begin{cases}\text{inside } r = 1+\cos\theta \\ \text{outside } r = 1\end{cases}$



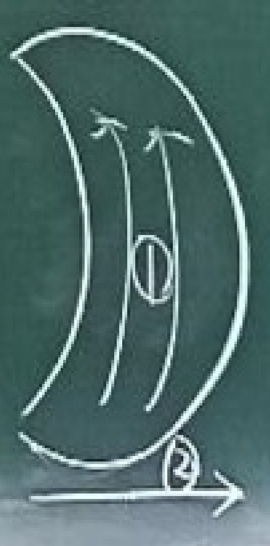
$r = 1+\cos\theta \Rightarrow \cos\theta = r-1$

$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0,\frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [\frac{-\pi}{2},0] \end{cases}$

(a) $r\,dr\,d\theta$: (usually preferred)

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

(b) $d\theta\,dr$ (usually not good)



$$I = \int_{r=1}^{2} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r,\theta)\, d\theta\, r\,dr$$

Remark

$r\,dr\,d\theta$ is usually a better choice than $d\theta\, r\,dr$ since most curves in polar coordinates appear naturally in the form $r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r\,dr$

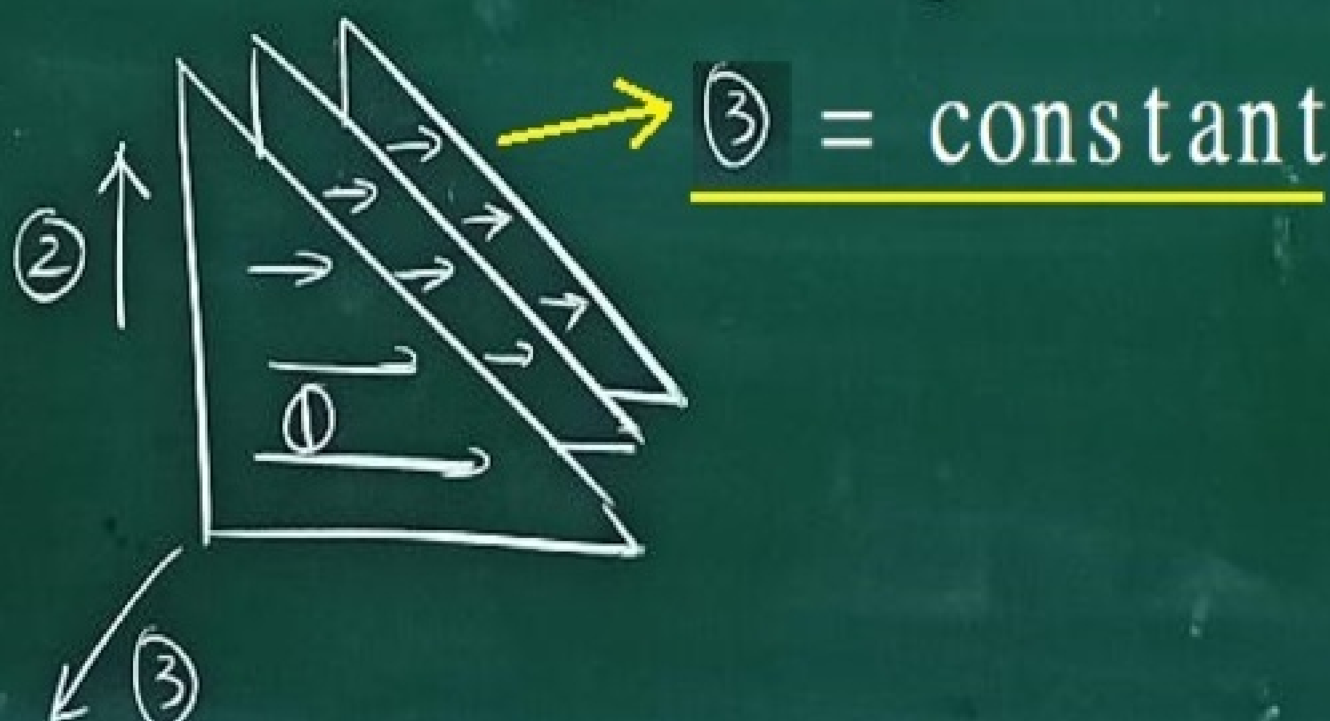
Triple integrals in Cartesian Coordinates

$$I = \iiint_D f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_{k=0}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

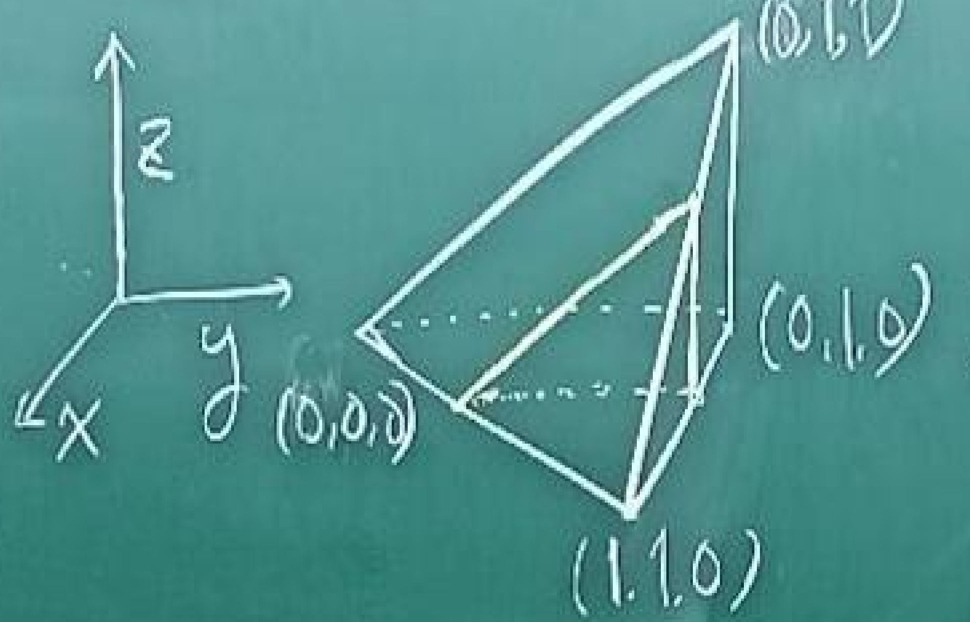
When $f(x,y,z) \equiv 1$, $I =$ volume of D

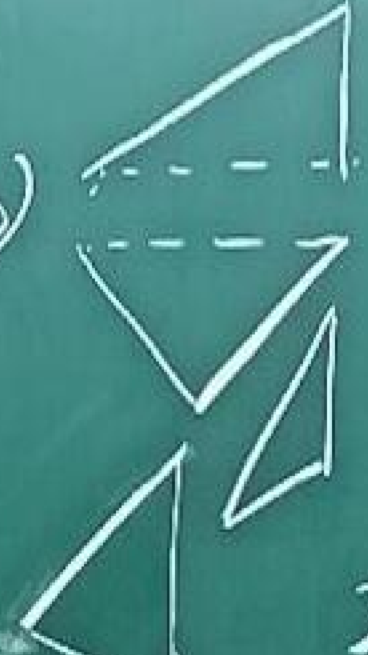
In Cartesian Coordinates

$$dV = dx\,dy\,dz = dy\,dz\,dx = \cdots\cdots$$



Eg 1 Find the volume of the domain enclosed by





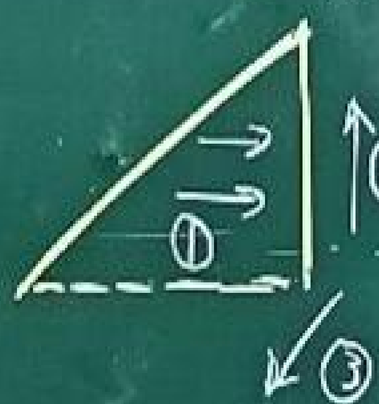
$x=0$ (A)

$z=0$ (B)

$y=1$ (C)

$x-y+z=0$ (D)

Method 1: $\overset{①}{dy}\,\overset{②}{dz}\,\overset{③}{dx}$ (cut along ③: $x=$ constant)



$$I=\int_{x=0}^{1}\int_{z=0\,(B)}^{z=1-x\,(C\cap D)}\int_{y=x+z\,(D)}^{y=1\,(C)} 1\,dy\,dz\,dx$$

① dy: need (C) and (D) in the form $y=f(x,z)$

$y=1$ $\quad y=x+z$

② dz: need (B) and $(C\cap D)$ in the form $z=g(x)$

$z=0$ $\quad z=1-x$ (eliminate y from (C),(D))

$$\therefore I = \int_{x=0}^{1}\int_{z=0}^{1-x}\int_{y=x-z}^{1} dy\,dz\,dx$$

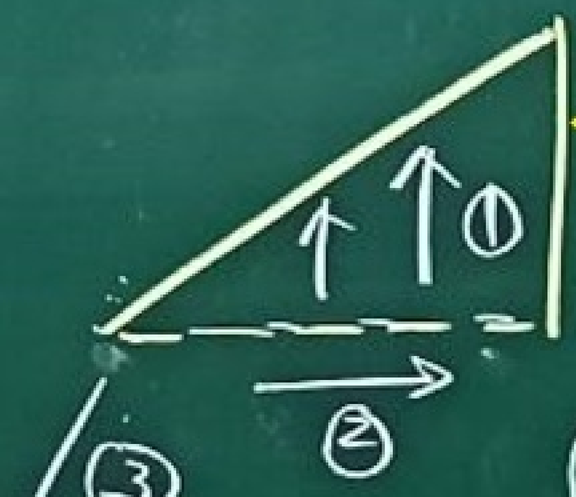
$$= \int_{x=0}^{1}\int_{z=0}^{1-x} (1-x+z)\,dz\,dx$$

$$= \int_{x=0}^{1} (1-x)z + \frac{z^2}{2}\Big|_{z=0}^{1-x}\,dx$$

$$= \int_{x=0}^{1} \frac{1}{2}(1-x)^2\,dx = \frac{1}{6}$$

Method 2: $dz\,dy\,dx$

① ② ③



Cut along ③ = const

Step ①:

$$\int_{(B)}^{(D)} 1\,dz = \int_{z=0}^{z=y-x} dz = y-x$$

Need (B), (D) as $z = F(x,y)$

Step ②

$$\int_{B\cap D}^{C} (y-x)\,dy = \int_{y=x}^{1} (y-x)\,dy$$

(Need $B\cap D$ and C as $y=G(x)$ by eliminating z ($=$ ①))

$$= \left(\frac{y^2}{2} - xy\right)\Big|_{y=x}^{1} = \frac{(1-x)^2}{2}$$

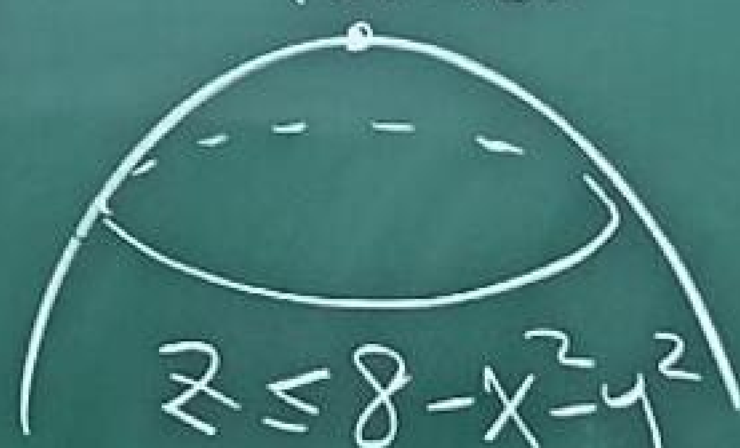
Step ③ $\displaystyle\int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{6}$

Other orders: $dx\,dy\,dz$, $dy\,dz\,dx$, ...

(Homework)

Eg2. Find the volume of

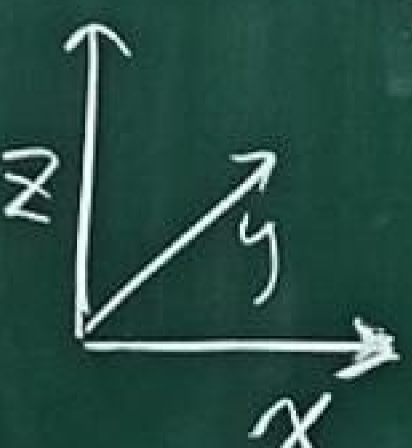
$$D=\{x^2+3y^2\le z\le 8-x^2-y^2\}$$



3 = ?

Not z!

$$\left\{\begin{array}{l} d\underset{z}{1}\,d\underset{y}{2}\,dx \\ \text{or} \\ \text{or } d\underset{z}{1}\,d\underset{x}{2}\,dy \end{array}\right.$$



are preferred (easier)

For example $dz\,dy\,dx$

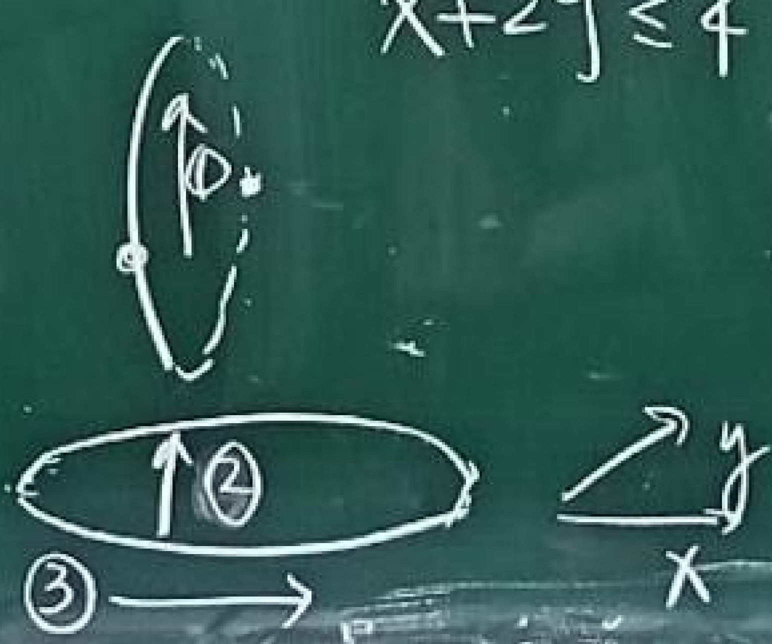
$$V = \iint_R \int_{z=x^2+3y^2}^{8-x^2-y^2} 1\, dz\, dA$$

$$R = \{(x,y) \mid x^2+3y^2 \le 8-x^2-y^2\}$$

(So that $\int_{z=x^2+y^2}^{8-x^2-y^2} dz$ is meaningful (positive))

$$= \{x^2+2y^2 \le 4\}$$

$$V = \iint_{x^2+2y^2 \le 4} 8-2x^2-4y^2\, dy\, dx$$



(Rm. Let $Y = 2y$

$dY = 2dy$

$$\iint_{x^2+Y^2 \le 4} 8-2(x^2+Y^2)\,\frac{dY}{2}\,dx$$

Polar coordinate integration)

$$\iint dy\,dx = \int_{x=?}^{?}\int_{y=?}^{?} dy\,dx$$

$$x^2+2y^2 \le 4$$

$$\Rightarrow -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

$$R = \left\{ \begin{array}{l} -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}} \\ -2 \le x \le 2 \end{array} \right\}$$

$$\therefore V = \int_{-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-y^2)\,dy\,dx$$

$$= \int_{-2}^{2} (8-2x^2)y - \frac{4y^3}{3} \Big|_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \frac{8}{3}(4-x^2)\sqrt{\frac{4-x^2}{2}}\,dx \quad \left(\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\,d\theta \end{array} \right)$$