

Integration in polar coordinate.

Eg 1 $\iint_R f(x, y) dA$

R = region enclosed by

$$\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases}$$



Both  ($dx dy$)

and  ($dy dx$)

are complicated.

In this example

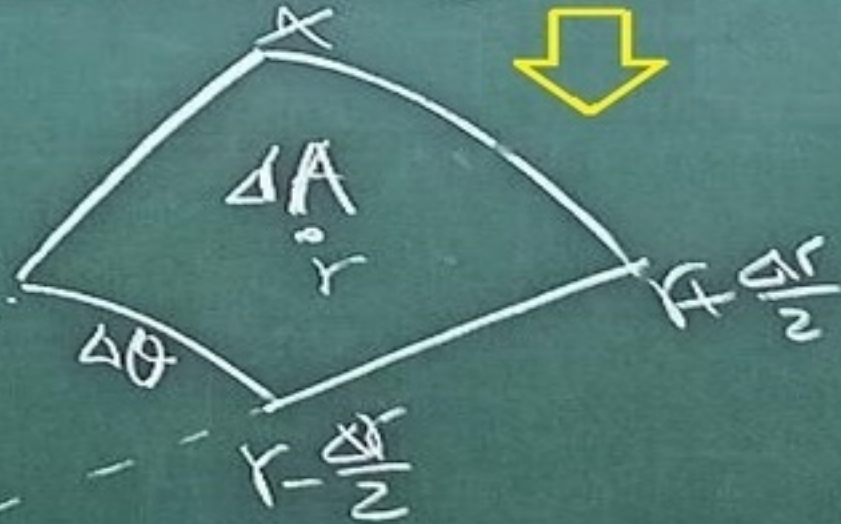
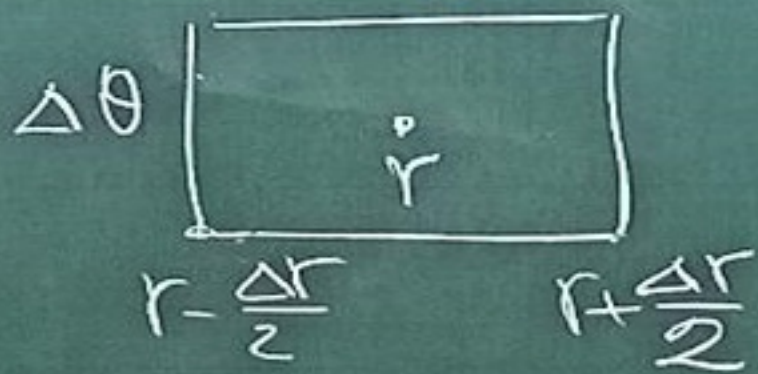
$$R = \left\{ 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \right\}$$

$$I = \int_{r=1}^2 \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} f(\underbrace{r \cos \theta}_x, \underbrace{r \sin \theta}_y) dA$$

$$dA \neq dr d\theta$$

Ans: $dA = r dr d\theta$

Why?



$$\begin{aligned} &= \pi \left(r + \frac{\Delta r}{2} \right)^2 \frac{\Delta \theta}{2\pi} - \pi \left(r - \frac{\Delta r}{2} \right)^2 \frac{\Delta \theta}{2\pi} \\ &= r \Delta r \Delta \theta \end{aligned}$$

Integration in Polar Coordinates

Ex 2 $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$



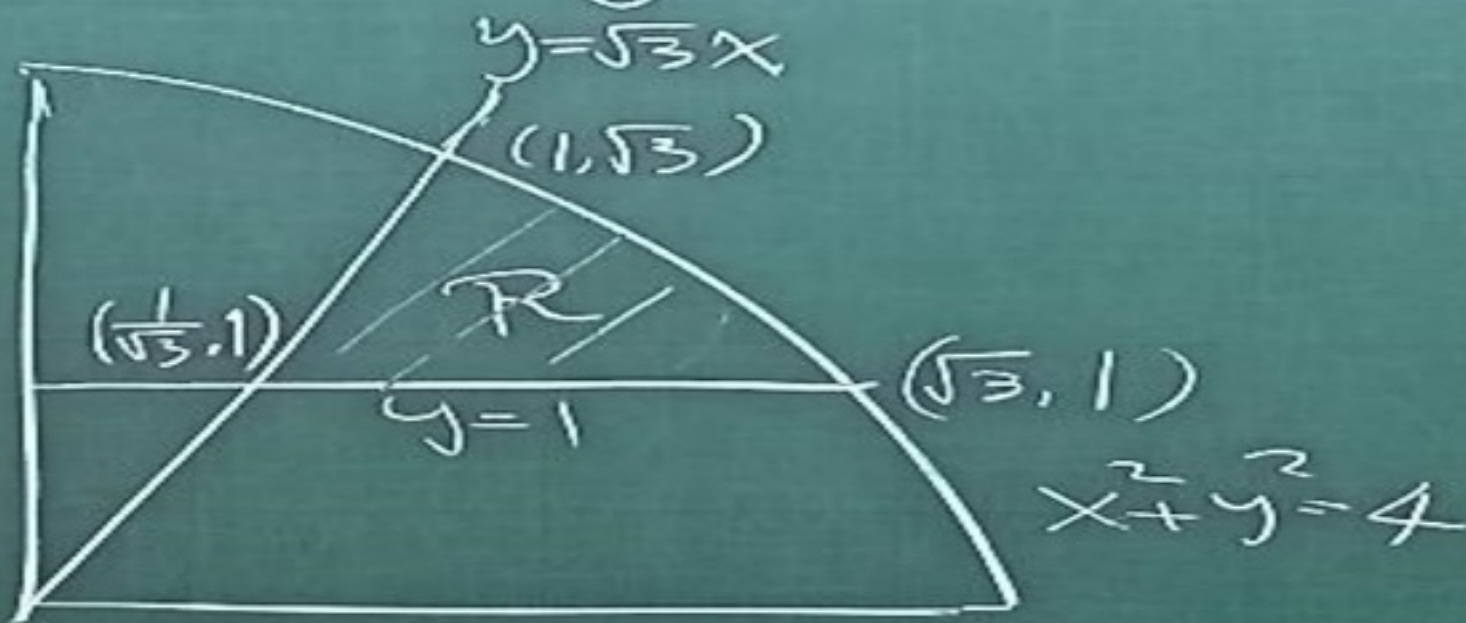
$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 r^2 (r dr d\theta)$$
$$= \left(\int_0^1 r^3 dr \right) \left(\int_0^{\frac{\pi}{2}} d\theta \right) = \frac{\pi}{8}$$

$\stackrel{dA}{=}$

Ex 3 Find the area enclosed by

$$\begin{cases} y = \sqrt{3}x \\ y = 1 \\ x^2 + y^2 = 4 \end{cases}$$

in 1st quadrant

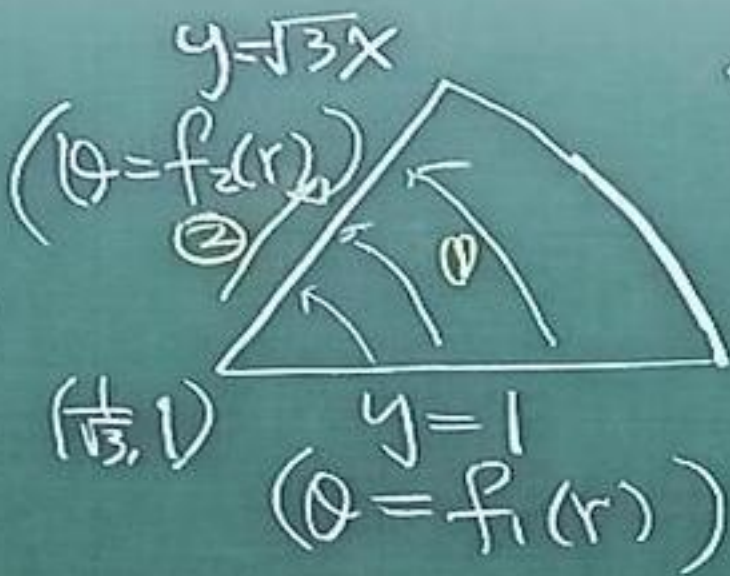


Sol: Method 1:

$$A = \text{Area of sector} - \text{Area of triangle}$$

$$= \pi \cdot 2^2 \cdot \frac{\frac{\pi}{6}}{2\pi} - \frac{1}{2} \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) \cdot 1$$

Method 2: Polar Coordinate



$$R = \left\{ \begin{array}{l} f_1(r) \leq \theta \leq f_2(r) \\ \frac{2}{\sqrt{3}} \leq r \leq 2 \end{array} \right\}$$

$$\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1^2}$$

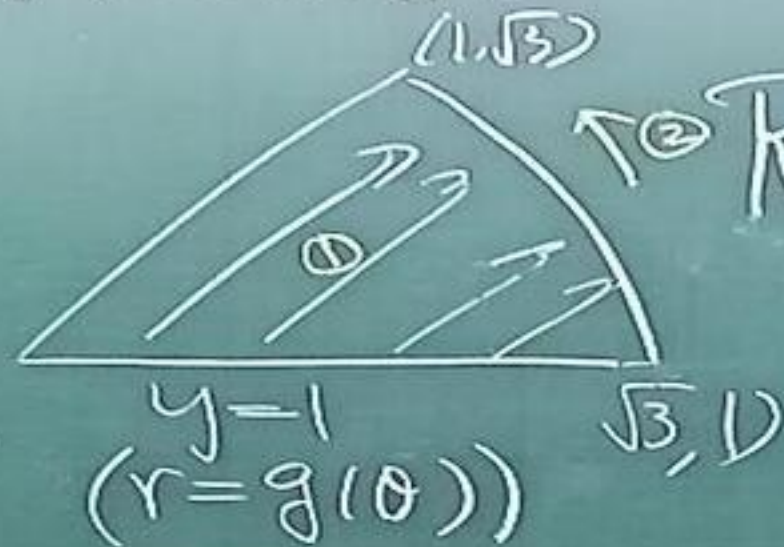
$$y = 1 \Leftrightarrow r \sin \theta = 1 \Rightarrow \theta = \sin^{-1}\left(\frac{1}{r}\right) = f_1(r)$$

$$y = \sqrt{3}x \Leftrightarrow \theta = \frac{\pi}{3} = f_2(r)$$

$$I = \int_{r=\frac{2}{\sqrt{3}}}^2 \int_{\theta=\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} d\theta r dr \rightarrow \text{difficult}$$

$$= \int_{\frac{2}{\sqrt{3}}}^2 r \left(\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{r}\right) \right) dr = ?$$

Method 3



$$R = \left\{ \begin{array}{l} g(\theta) \leq r \leq 2 \\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3} \end{array} \right\}$$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
 $\tan^{-1}(\sqrt{3})$

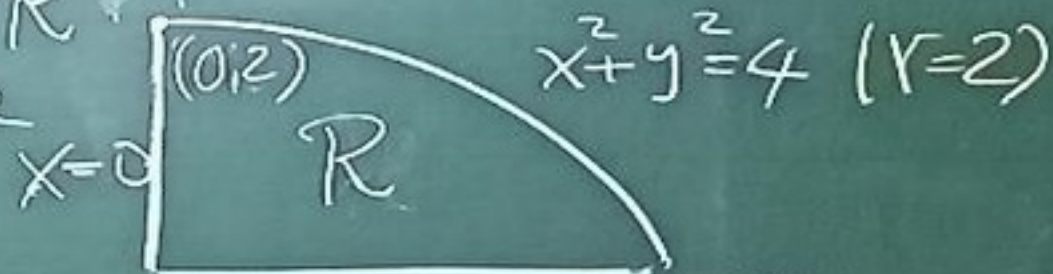
$$y=1 \Leftrightarrow r \sin \theta = 1 \Rightarrow r = \frac{1}{\sin \theta} = g(\theta)$$

$$I = \int_{\theta = \frac{\pi}{6}}^{\frac{\pi}{3}} \int_{r = \frac{1}{\sin \theta}}^2 r \, dr \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(2 - \frac{1}{2 \sin^2 \theta} \right) d\theta$$

$$= \left(2\theta + \frac{\cot \theta}{2} \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{1}{2} \left(\frac{1}{\sqrt{3}} - \sqrt{3} \right)$$

Eg 4. Evaluate $\iint_R f(x,y) dA$ in polar coordinate where $x^2 + y^2 = 4$ ($r=2$)

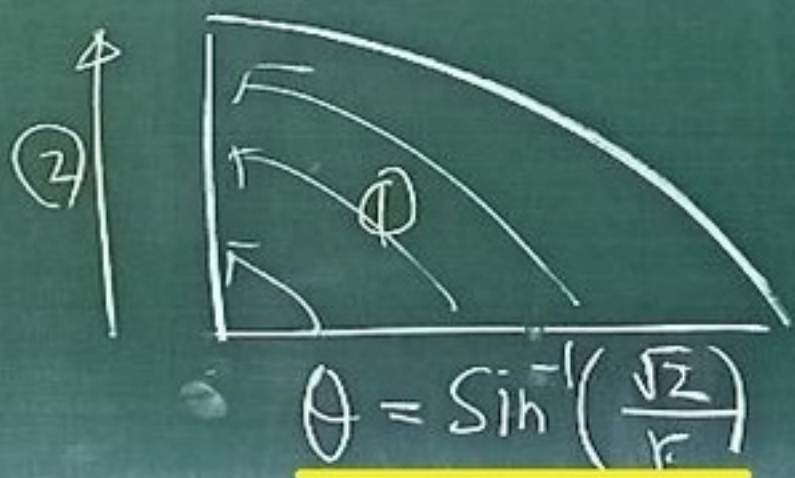


Sol. Method (a) $r dr d\theta$



$$I = \int_{\theta = \frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r = \frac{2}{\sin \theta}}^2 f(r \cos \theta, r \sin \theta) r dr d\theta$$

Method (b) $d\theta r dr$



$$I = \int_{r = \sqrt{2}}^2 \int_{\theta = \sin^{-1}(\frac{\sqrt{2}}{r})}^{\frac{\pi}{2}} f(r \cos \theta, r \sin \theta) d\theta r dr$$

Eg 5 Evaluate $\iint_R g(r, \theta) dA$

where R is the region $\begin{cases} \text{inside } r=1+\cos\theta \\ \text{outside } r=1 \end{cases}$



$$r = 1 + \cos\theta \Rightarrow \cos\theta = r - 1$$

$$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0, \frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [\frac{\pi}{2}, 0] \end{cases}$$

(a) $r dr d\theta$
(usually preferred)



$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r, \theta) r dr d\theta$$

(b) $d\theta r dr$



$$I = \int_{r=1}^{\cos^{-1}(r-1)} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r, \theta) d\theta r dr$$

(usually not good)

Remark

$r \, dr \, d\theta$ is usually a better choice than $d\theta \, r \, dr$ since most curves in polar coordinates appear naturally in the form

$r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r \, dr$

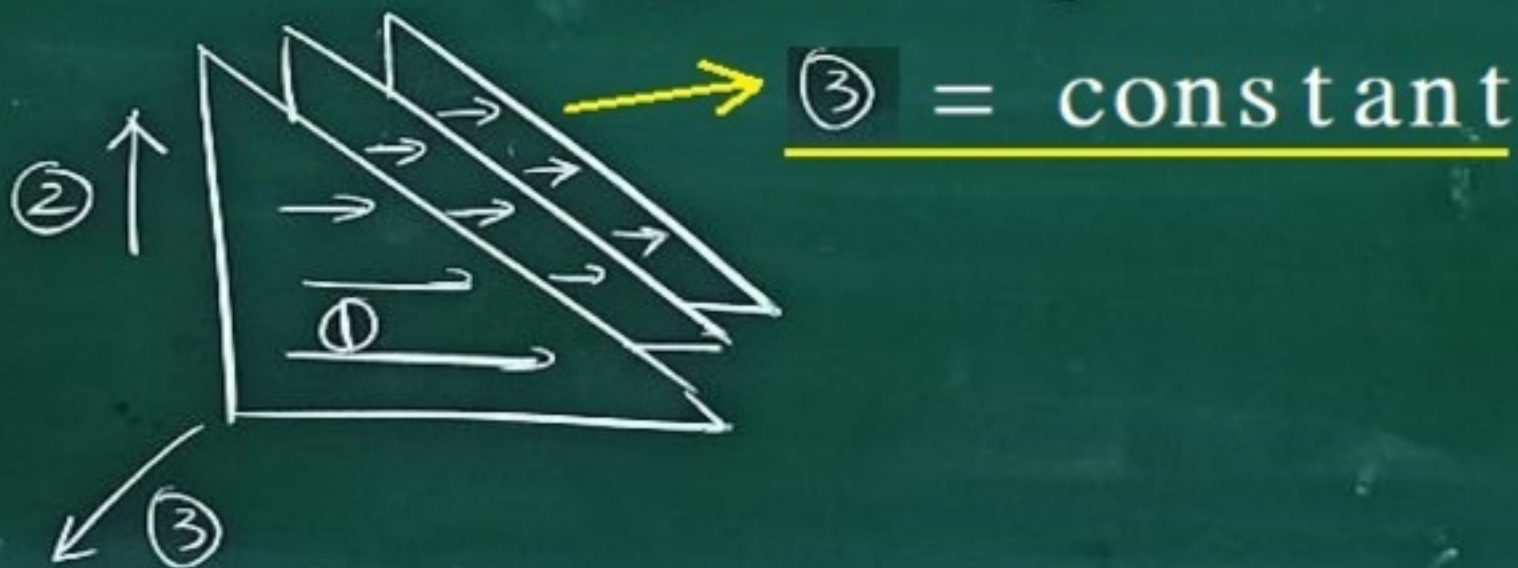
Triple integrals in Cartesian Coordinates

$$I = \iiint_D f(x, y, z) dV = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta V_k$$

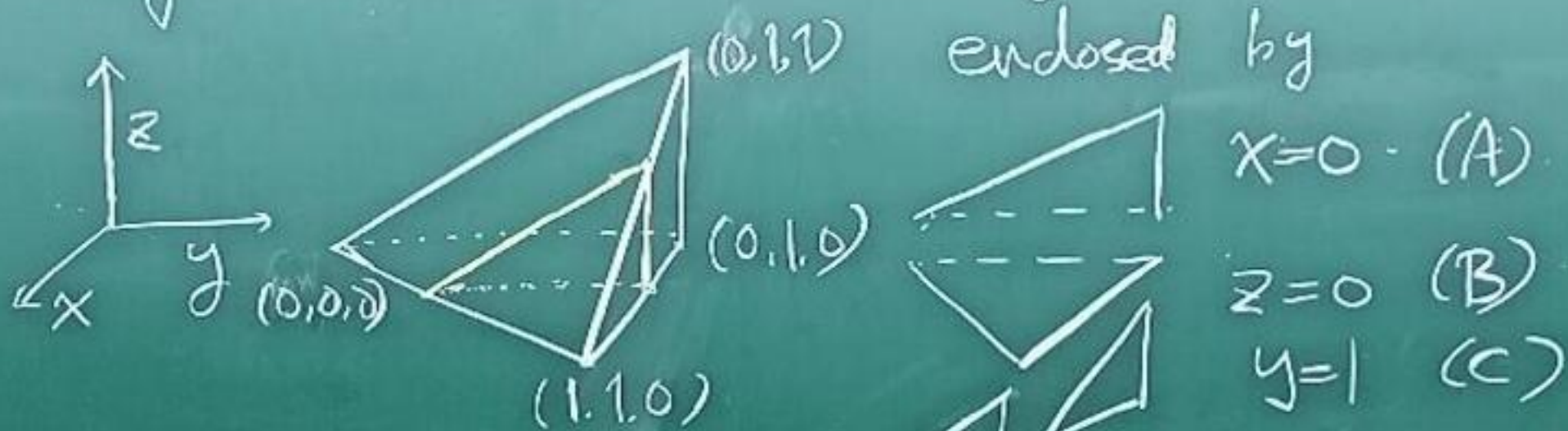
When $f(x, y, z) \equiv 1$, $I = \text{volume of } D$

In Cartesian Coordinates

$$dV = dx dy dz = dy dz dx = \dots$$



Eg 1 Find the volume of the domain



Method 1: $\overset{\textcircled{1}}{dy} \overset{\textcircled{2}}{dz} \overset{\textcircled{3}}{dx}$ (cut along ③: $x = \text{constant}$)

$$I = \int_{x=0}^1 \int_{z=0}^{z=1-x} \int_{y=x+z}^{y=1} 1 \, dy \, dz \, dx$$

(C ∩ D)

(B)

(D)

① dy : need (C) and (D) in the form $y = f(x, z)$

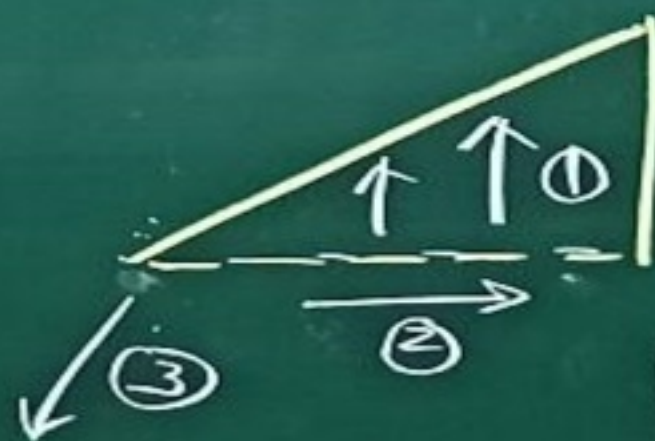
$y=1$ $y=x+z$

② dz : need (B) and (C ∩ D) in the form $z = g(x)$

$z=0$ $z=1-x$ (eliminate y from (C), (D))

$$\begin{aligned}
 \therefore I &= \int_{x=0}^1 \int_{z=0}^{1-x} \int_{y=x-z}^1 dy dz dx \\
 &= \int_{x=0}^1 \int_{z=0}^{1-x} (1-x+z) dz dx \\
 &= \int_{x=0}^1 \left((1-x)z + \frac{z^2}{2} \right) \Big|_{z=0}^{1-x} dx \\
 &= \int_{x=0}^1 \frac{1}{2}(1-x)^2 dx = \frac{1}{6}
 \end{aligned}$$

Method 2: $dz dy dx$



Cut along ③ = const

Step ①:

$$\begin{aligned}
 \text{(D)} \quad 1 dz &= \int_{z=0}^{z=y-x} dz \\
 \text{(B)} \quad &= \int_{z=0}^{z=y-x} dz
 \end{aligned}$$

Need (B), (D) as $z = F(x, y)$

Step ②

$$\int_{B \cap D}^C (y-x) dy = \int_{y=x}^1 (y-x) dy$$

(Need $B \cap D$ and C as $y=G(x)$
by eliminating z (= ①))

$$= \int_{y=x}^1 \left(\frac{y^2}{2} - xy \right) dy = \frac{(1-x)^2}{2}$$

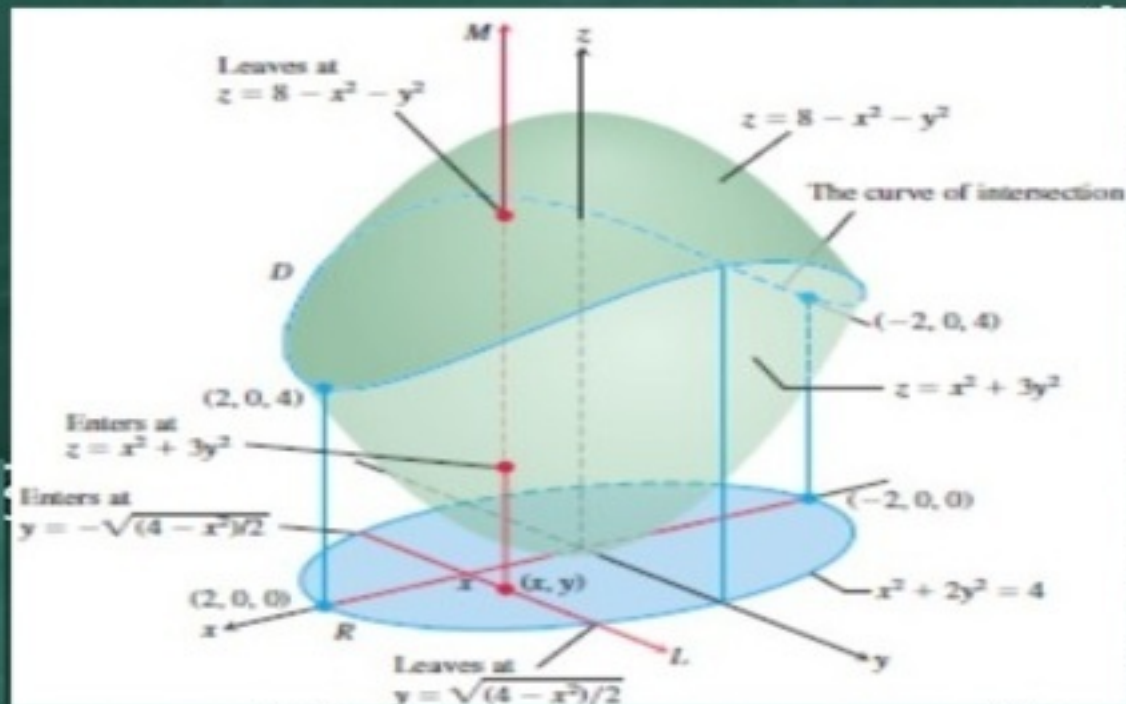
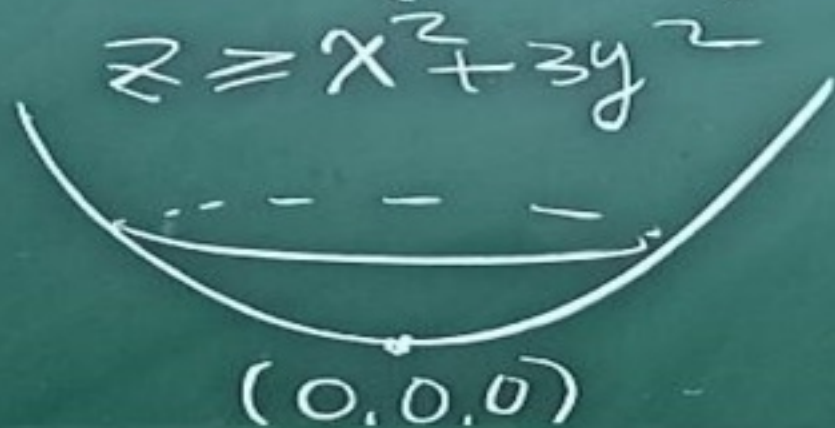
Step ③ $\int_0^1 \frac{(1-x)^2}{2} dx = \frac{1}{6}$

Other orders: $\frac{dx dy dz}{dy dz dx}, \dots$

(Homework)

Ex 2. Find the volume of

$$D = \{ x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2 \}$$



③ = ?

Not \mathbb{R} !

$\int \int \mathbb{R} \, dx \, dy$

or $\int \int \mathbb{R} \, dx \, dy$
are preferred (easier)

For example $dz dy dx$

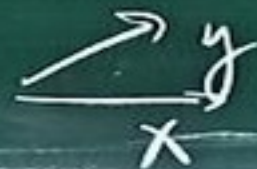
$$V = \iiint_R \int_{z=x^2+3y^2}^{8-x^2-y^2} 1 dz dA$$

$$R = \{(x, y) \mid x^2 + 3y^2 \leq 8 - x^2 - y^2\}$$

(So that $\int_{z=x^2+y^2}^{8-x^2-y^2} dz$ is meaningful)
(positive)

$$= \{x^2 + 2y^2 \leq 4\}$$

$$V = \iint_{x^2+2y^2 \leq 4} (8 - 2x^2 - 4y^2) dy dx$$



Rm. Let $Y = \sqrt{2}y$
 $dY = \sqrt{2} dy$
 $\iint_{x^2+Y^2 \leq 4} (8 - 2(x^2 + Y^2)) \frac{dY}{\sqrt{2}} dx$
Polar coordinate integration

$$\iiint dy dx = \int_{x=?}^? \int_{y=?}^? dy dx$$

$$x^2 + 2y^2 \leq 4$$

$$\Rightarrow -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}$$

$$R = \left\{ \begin{array}{l} -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}} \\ -2 \leq x \leq 2 \end{array} \right\}$$

$$\therefore V = \int_{-2}^2 \int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - y^2) dy dx$$

$$= \int_{-2}^2 \left((8 - 2x^2)y - \frac{4y^3}{3} \right) \Big|_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^2 \frac{8}{3}(4-x^2) \sqrt{\frac{4-x^2}{2}} dx \quad \left(\begin{array}{l} x = 2 \sin \theta \\ dx = 2 \cos \theta d\theta \end{array} \right)$$