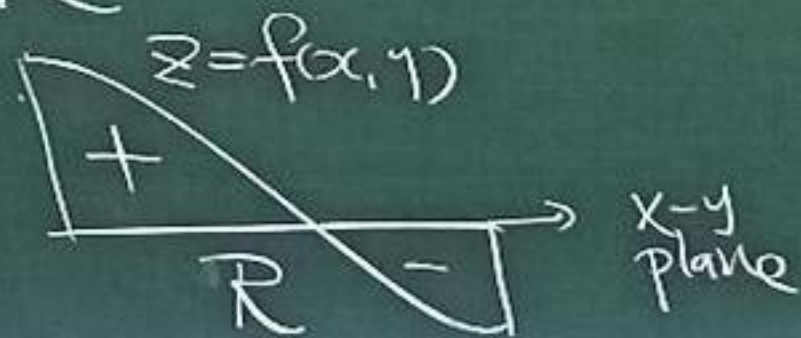
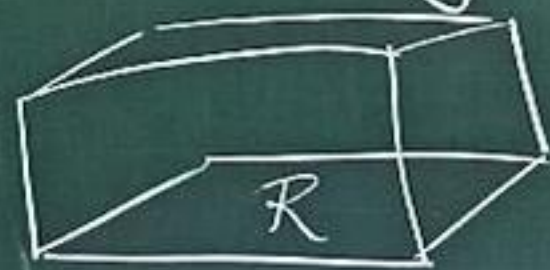


Double Integral

Def: Let $R = [a, b] \times [c, d]$
 $= \{ a \leq x \leq b, c \leq y \leq d \}$

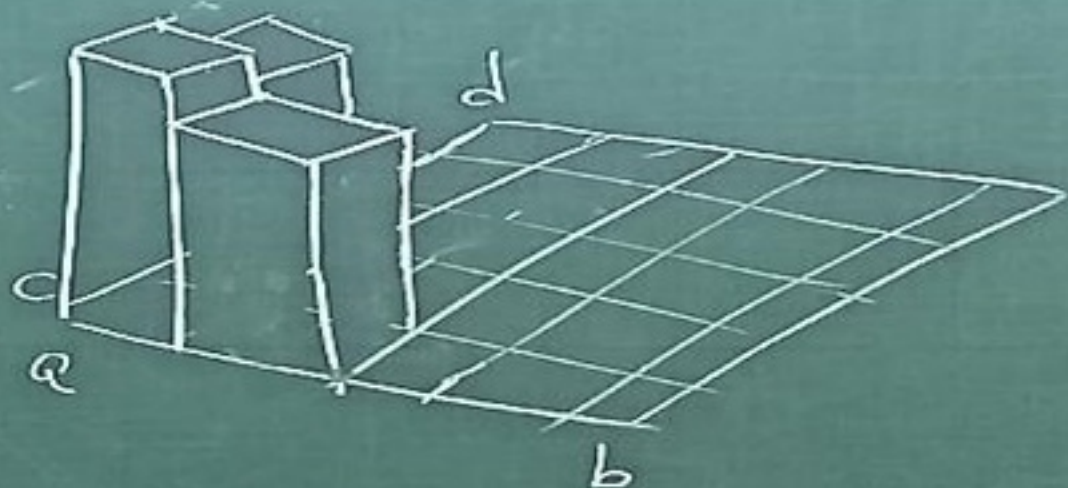
$$\iint_R f(x, y) dA$$


= Signed Volume between
 $z = f(x, y)$ graph and x - y plane
over the region R .



In other words,

$$\iint_R f(x, y) dA = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$



R_k :  - kth rectangle
Area $\equiv \Delta A_k$

$$P = \left\{ \begin{array}{l} a = \bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_n = b \\ c = \bar{y}_0 < \bar{y}_1 < \dots < \bar{y}_n = d \end{array} \right\}$$

$$\|P\| = \max_{\text{all } i, j} \left\{ \begin{array}{l} \Delta x_i = \bar{x}_i - \bar{x}_{i-1} \\ \Delta y_j = \bar{y}_j - \bar{y}_{j-1} \end{array} \right\}$$

$(x_k, y_k) \in R_k$

In general, if $f(x, y)$ is continuous on R , then

$\iint_R f(x, y) dA$ exists.

(f is integrable on R)

Fubini's Theorem.

If f is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

General Region R

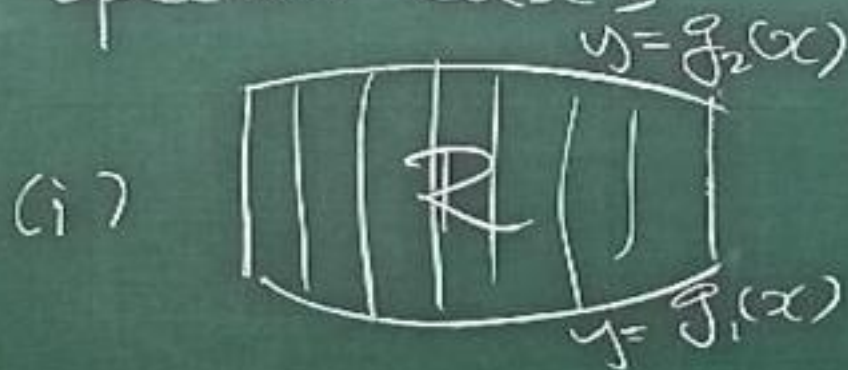


$$\iint_R f(x, y) dA$$

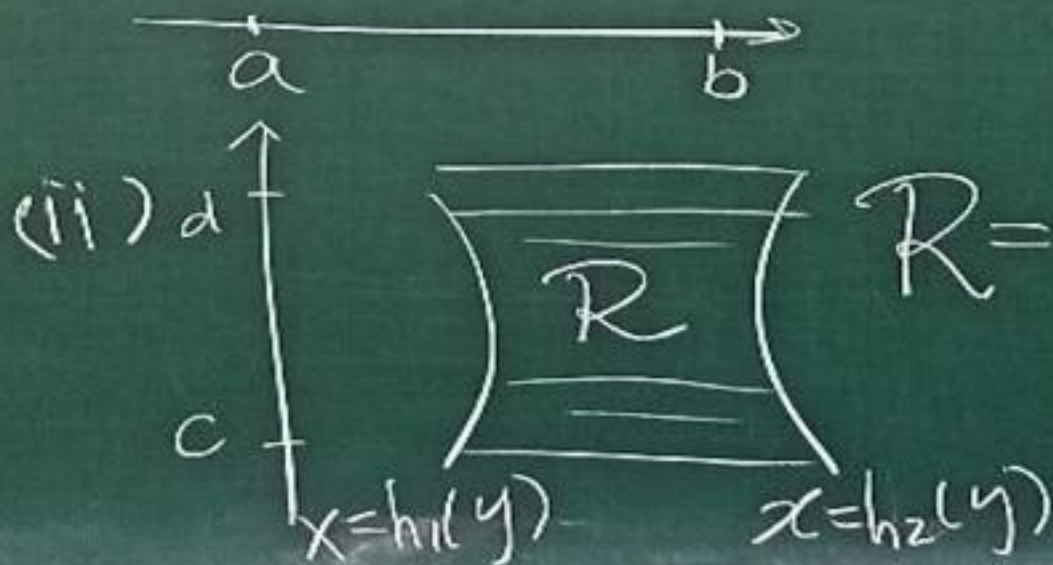
$$= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

$R_k \subset R$

Special cases



$$R = \left\{ \begin{array}{l} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{array} \right\}$$



$$R = \left\{ \begin{array}{l} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{array} \right\}$$

Fubini's Theorem (Strong form)

If f is cont. on R and

$$(i) R = \{a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

$$\text{Then } \iint_R f(x, y) dA = \int_{x=a}^b \int_{y=g_1(x)}^{g_2(x)} f(x, y) dy dx$$

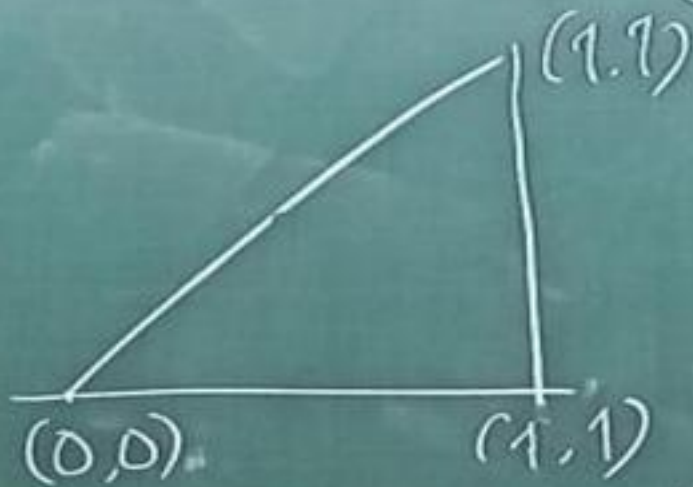
or

$$(ii) R = \{c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$



$$\text{Then } \iint_R f(x, y) dA = \int_{y=c}^d \int_{x=h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Ex 1 R = region enclosed by

$$x=y, x=1 \text{ and } y=0$$



$$\iint_R \frac{\sin x}{x} dA = ?$$

Sol: $R = \left\{ 0 \leq x \leq 1, 0 \leq y \leq x \right\}^{(i)}$ 
 $= \left\{ 0 \leq y \leq 1, y \leq x \leq 1 \right\}^{(ii)}$ 

$$(ii) = \int_{y=0}^1 \int_{x=y}^1 \frac{\sin x}{x} dx dy = ?$$

$$(i) = \int_{x=0}^1 \int_{y=0}^x \frac{\sin x}{x} dy dx$$

$$= \int_0^1 \left(\frac{\sin x}{x} \int_{y=0}^x 1 dy \right) dx = -\cos x \Big|_0^1 = 1 - \cos 1$$

Eg2: Find the area of the region enclosed by

$$\begin{cases} y=0 \\ y=4x-2 \\ x=\frac{y^2}{4} \end{cases} \quad (y > 0)$$

Sol: $A = \iint_R 1 \, dA$



(I): exercise.

$$(II) = \int_{x=0}^1 \int_{y=0}^{2\sqrt{x}} 1 \, dy \, dx - \frac{1}{2} \cdot \frac{1}{2} \cdot 2$$

$$= \int_0^1 2\sqrt{x} \, dx - \frac{1}{2} = \frac{5}{6}$$

$$(III) = \int_{y=0}^2 \int_{x=\frac{y^2}{4}}^{\frac{y+2}{4}} 1 \, dx \, dy$$

$$= \int_{y=0}^2 \left(\frac{y+2}{4} - \frac{y^2}{4} \right) dy$$

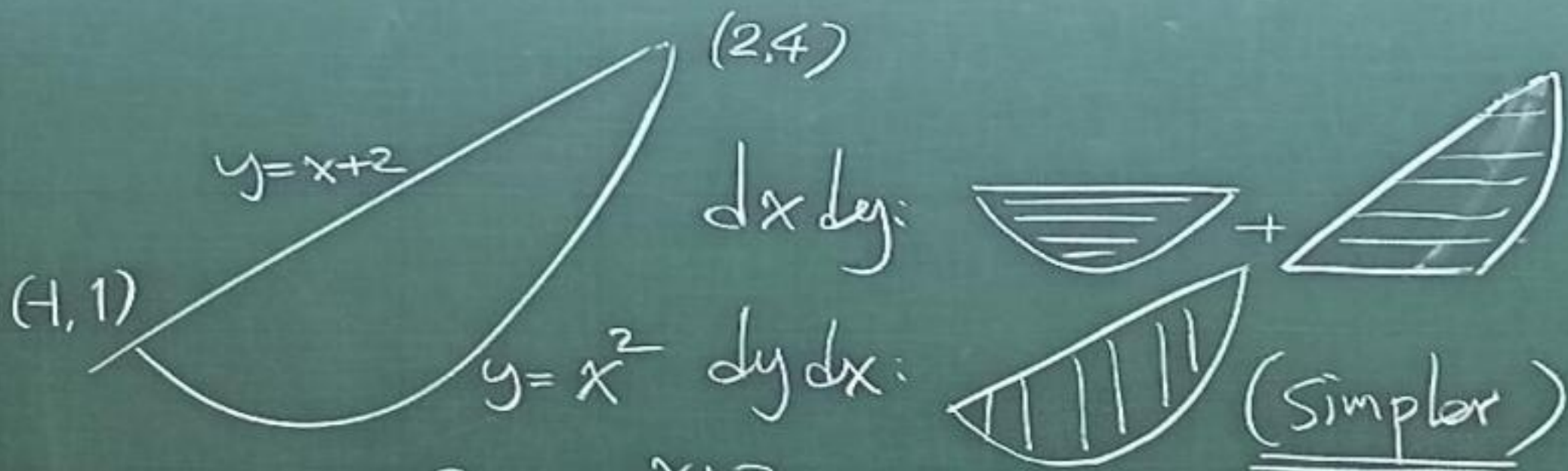
$$= \int_0^2 -\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2} \, dy$$

$$= -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2} \Big|_{y=0}^2$$

$$= \frac{5}{6}$$

Eg 3. Find area of the region

enclosed by $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

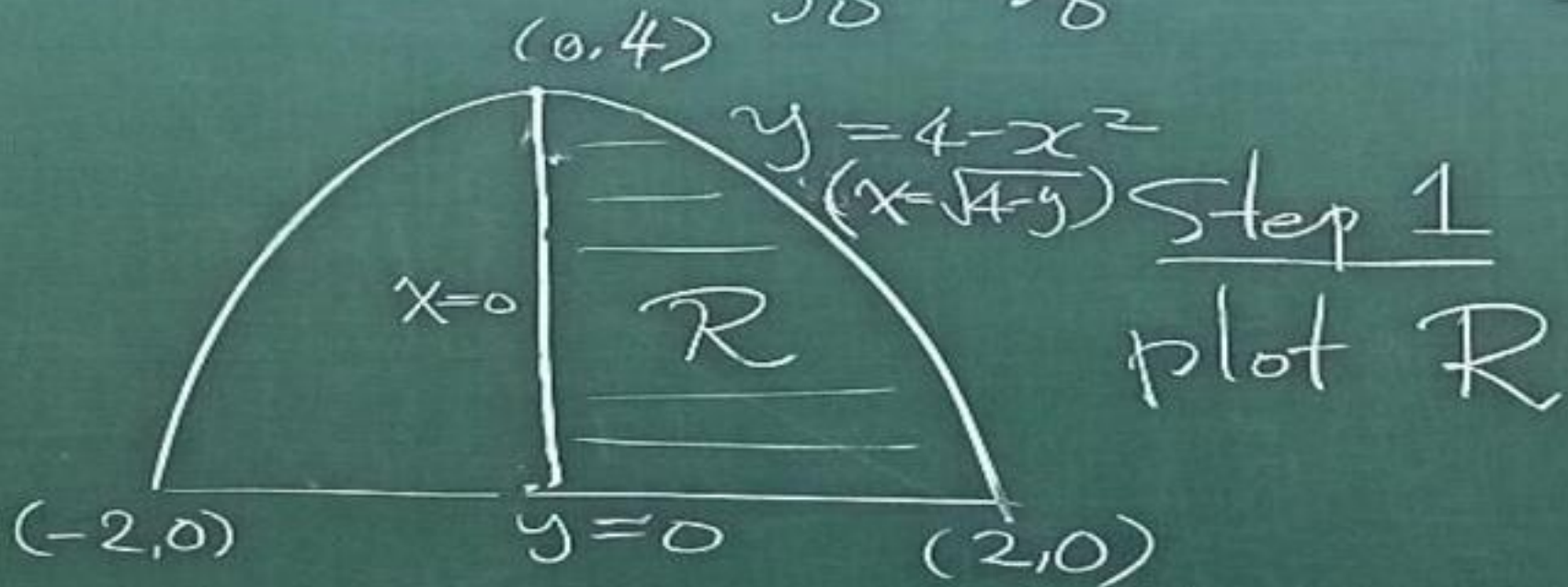


$$A = \int_{x=-1}^2 \int_{y=x^2}^{x+2} dy dx = \int_{-1}^2 (x+2-x^2) dx$$
$$= \left. \frac{x^2}{2} + 2x - \frac{x^3}{3} \right|_{-1}^2 = \frac{9}{2}$$

Ex 4: $\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} dy dx = ?$

Sol: Try $dx dy$ instead.

Note: $\neq \int_0^2 \int_0^{4-x^2} \dots dx dy$



Step 2: Find limits of integration for dx

left: $x = 0$

Right: $y = 4 - x^2$, $x = \sqrt{4 - y}$

Step 3

$$A = \int_{y=0}^4 \int_{x=0}^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} dx dy$$

$$= \int_{y=0}^4 \frac{e^{2y}}{4-y} \int_{x=0}^{\sqrt{4-y}} x dx dy$$

$$= \int_{y=0}^4 \frac{e^{2y}}{4-y} \cdot \left(\frac{x^2}{2} \Big|_0^{\sqrt{4-y}} \right) dy$$

$$= \frac{1}{2} \int_0^4 e^{2y} dy$$

$$= \frac{1}{4} e^{2y} \Big|_0^4 = \frac{e^8 - 1}{4}$$

Integration in polar coordinate.

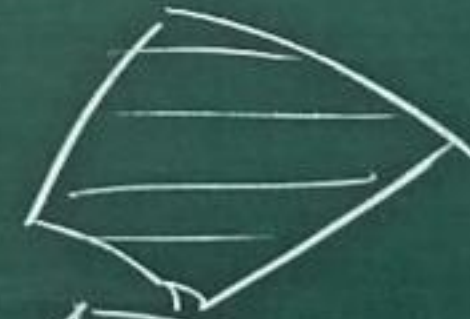
Eg 5 $\iint_R f(x, y) dA$

R = region enclosed by

$$\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases}$$



Both



($dx dy$)

and



($dy dx$)

are complicated.

In this example

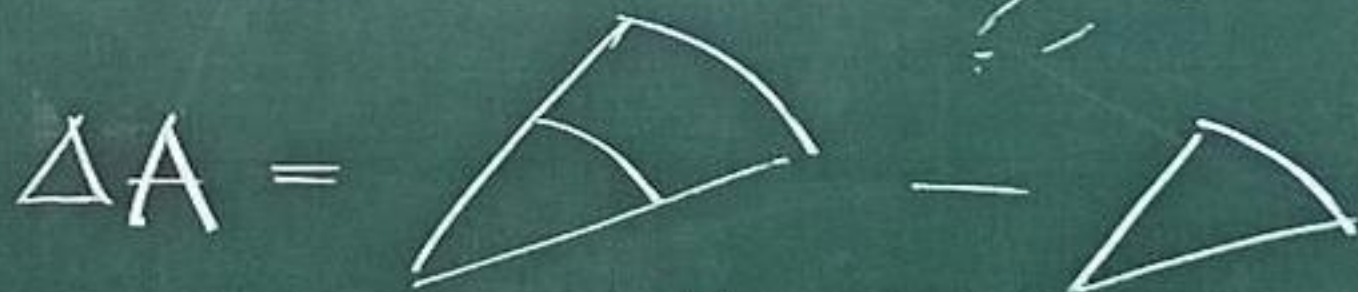
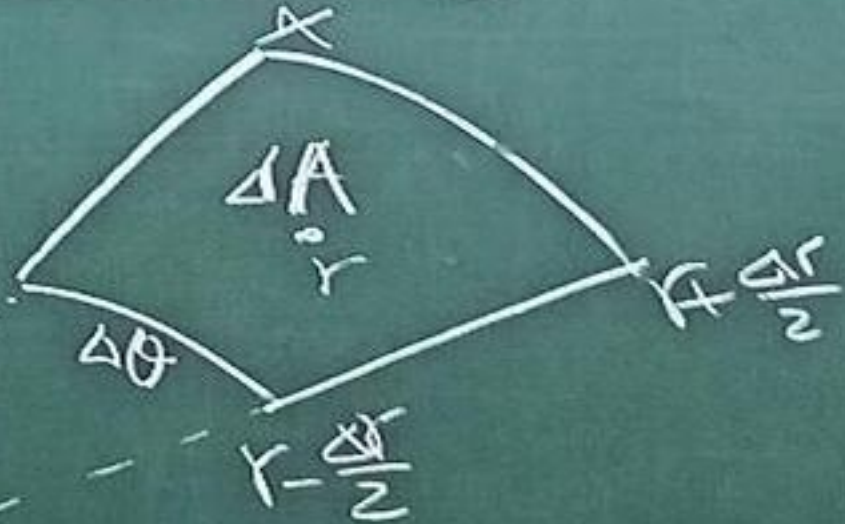
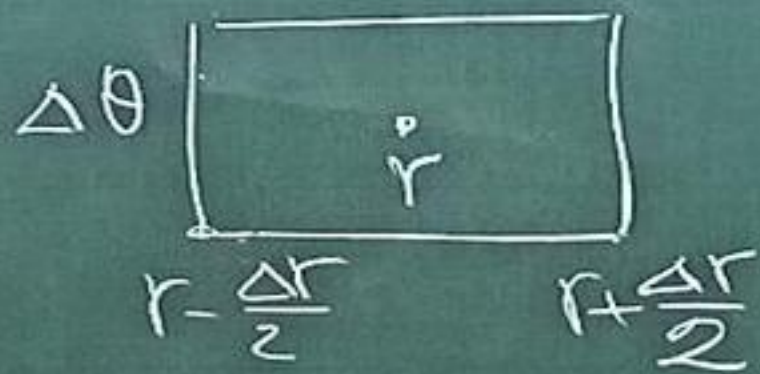
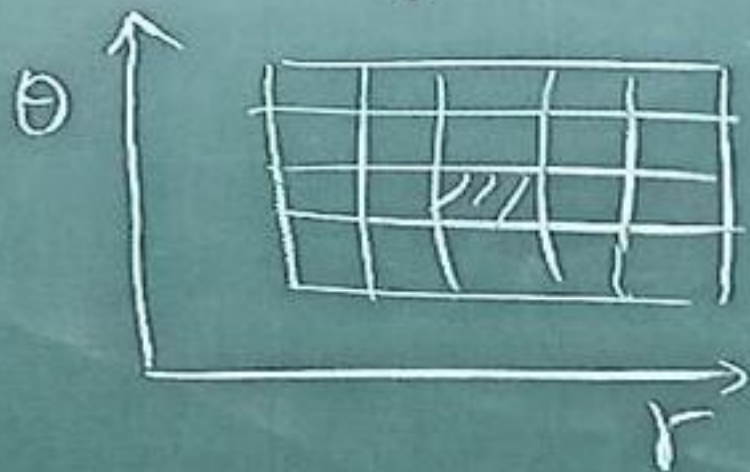
$$R = \left\{ 1 \leq r \leq 2, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3} \right\}$$

$$I = \int_{\theta = \frac{\pi}{4}}^{\frac{\pi}{3}} f(\underbrace{r \cos \theta}_x, \underbrace{r \sin \theta}_y) dA$$

$$dA \neq dr d\theta$$

Ans: $dA = r dr d\theta$

Why?



$$\begin{aligned} \Delta A &= \pi \left(r + \frac{\Delta r}{2} \right)^2 \frac{\Delta \theta}{2\pi} - \pi \left(r - \frac{\Delta r}{2} \right)^2 \frac{\Delta \theta}{2\pi} \\ &= r \Delta r \Delta \theta \end{aligned}$$

Integration in Polar Coordinates

Ex 2 $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$



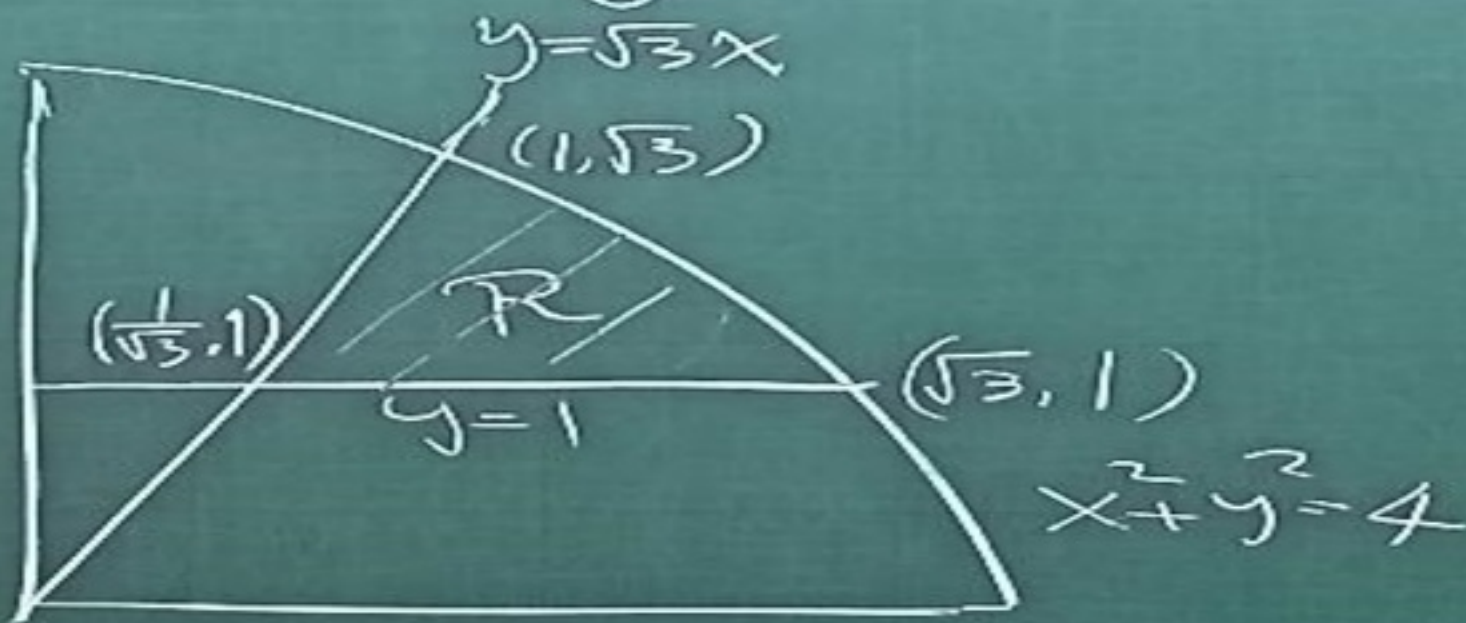
$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 r^2 (r dr d\theta)$$
$$= \left(\int_0^1 r^3 dr \right) \left(\int_0^{\frac{\pi}{2}} d\theta \right) = \frac{\pi}{8}$$

$\stackrel{dA}{=}$

Ex 3 Find the area enclosed by

$$\begin{cases} y = \sqrt{3}x \\ y = 1 \\ x^2 + y^2 = 4 \end{cases}$$

in 1st quadrant

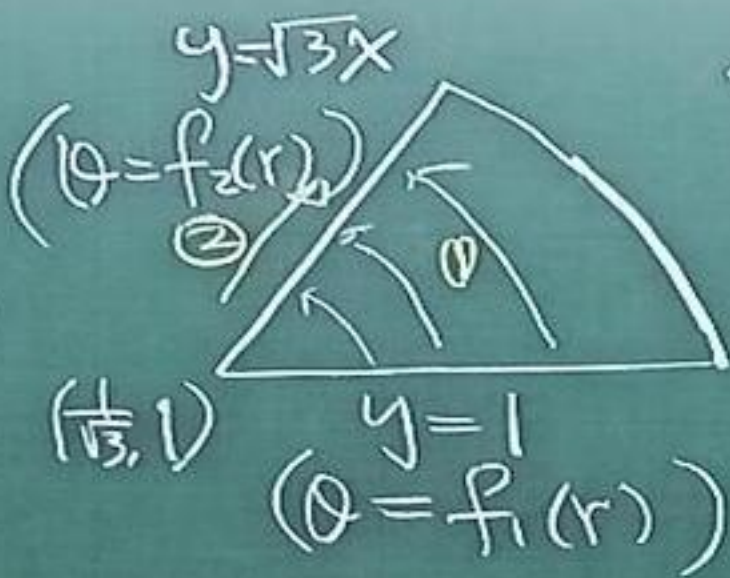


Sol: Method 1:

$$A = \text{Area of sector} - \text{Area of triangle}$$

$$= \pi \cdot 2^2 \cdot \frac{\pi/6}{2\pi} - \frac{1}{2} \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) \cdot 1$$

Method 2: Polar Coordinate



$$R = \left\{ \begin{array}{l} f_1(r) \leq \theta \leq f_2(r) \\ \frac{2}{\sqrt{3}} \leq r \leq 2 \end{array} \right\}$$

$$\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1^2}$$

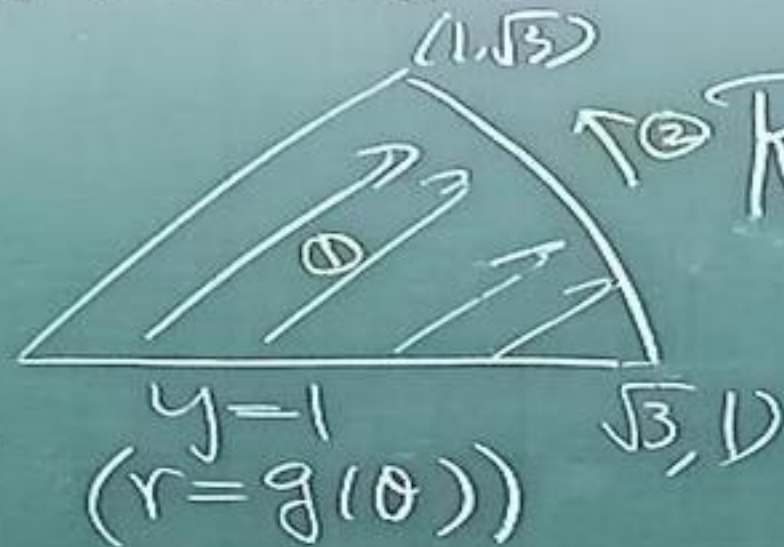
$$y = 1 \Leftrightarrow r \sin \theta = 1 \Rightarrow \theta = \sin^{-1}\left(\frac{1}{r}\right) = f_1(r)$$

$$y = \sqrt{3}x \Leftrightarrow \theta = \frac{\pi}{3} = f_2(r)$$

$$I = \int_{r=\frac{2}{\sqrt{3}}}^2 \int_{\theta=\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} d\theta r dr \rightarrow \text{difficult}$$

$$= \int_{\frac{2}{\sqrt{3}}}^2 r \left(\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{r}\right) \right) dr = ?$$

Method 3



$$R = \left\{ \begin{array}{l} g(\theta) \leq r \leq 2 \\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3} \end{array} \right\}$$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
 $\tan^{-1}(\sqrt{3})$

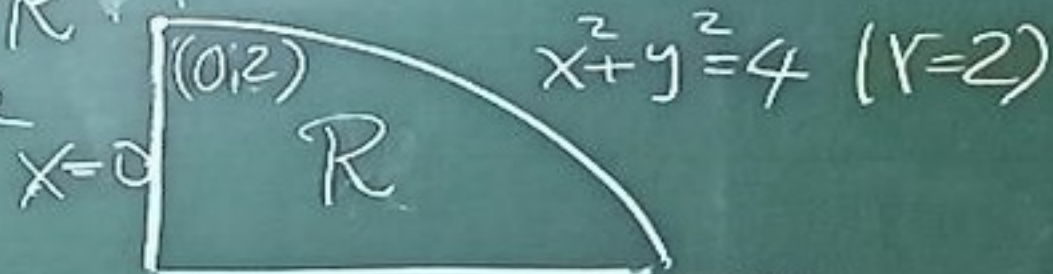
$$y=1 \Leftrightarrow r \sin \theta = 1 \Rightarrow r = \frac{1}{\sin \theta} = g(\theta)$$

$$I = \int_{\theta = \frac{\pi}{6}}^{\frac{\pi}{3}} \int_{r = \frac{1}{\sin \theta}}^2 r \, dr \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(2 - \frac{1}{2 \sin^2 \theta} \right) d\theta$$

$$= \left(2\theta + \frac{\cot \theta}{2} \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{1}{2} \left(\frac{1}{\sqrt{3}} - \sqrt{3} \right)$$

Eg 4. Evaluate $\iint_R f(x,y) dA$ in polar coordinate where $x^2 + y^2 = 4$ ($r=2$)

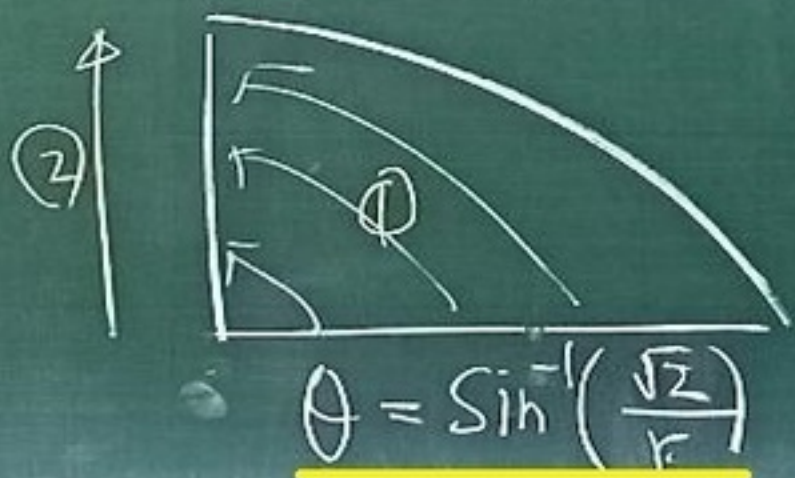


Sol. Method (a) $r dr d\theta$



$$I = \int_{\theta = \frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r = \frac{2}{\sin \theta}}^2 f(r \cos \theta, r \sin \theta) r dr d\theta$$

Method (b) $d\theta r dr$



$$I = \int_{r = \sqrt{2}}^2 \int_{\theta = \sin^{-1}(\frac{\sqrt{2}}{r})}^{\frac{\pi}{2}} f(r \cos \theta, r \sin \theta) d\theta r dr$$

Eg 5 Evaluate $\iint_R g(r, \theta) dA$

where R is the region $\begin{cases} \text{inside } r=1+\cos\theta \\ \text{outside } r=1 \end{cases}$



$$r=1+\cos\theta \Rightarrow \cos\theta = r-1$$

$$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0, \frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [-\frac{\pi}{2}, 0] \end{cases}$$

(a) $r dr d\theta$
(usually preferred)



$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r, \theta) r dr d\theta$$

(b) $d\theta r dr$



$$I = \int_{r=1}^{\cos^{-1}(r-1)} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r, \theta) d\theta r dr$$

(usually not good)

Remark

$r \, dr \, d\theta$ is usually a better choice than $d\theta \, r \, dr$ since most curves in polar coordinates appear naturally in the form

$r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r \, dr$