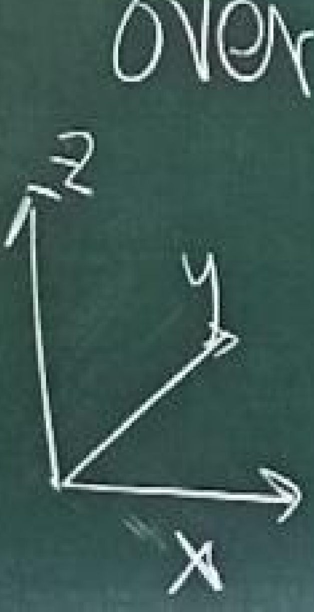
# Double Integral

Def: Let $R = [a,b] \times [c,d]$

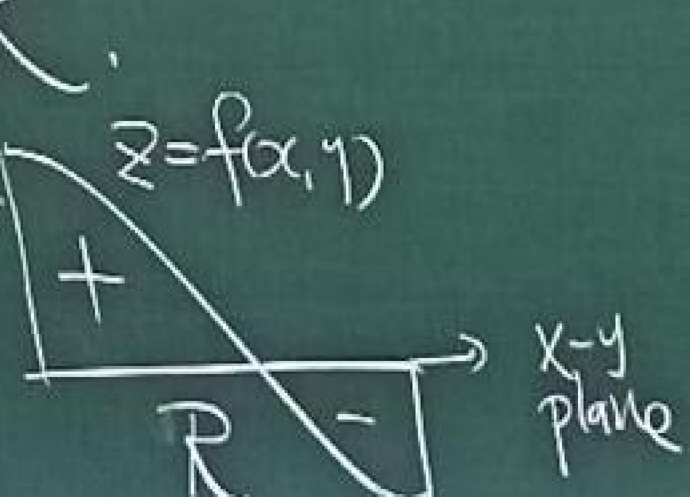
$= \{ a \le x \le b,\ c \le y \le d \}$

$$\iint_R f(x,y)\, dA$$

= Signed Volume between $z = f(x,y)$ graph and $x$-$y$ plane over the region $R$.

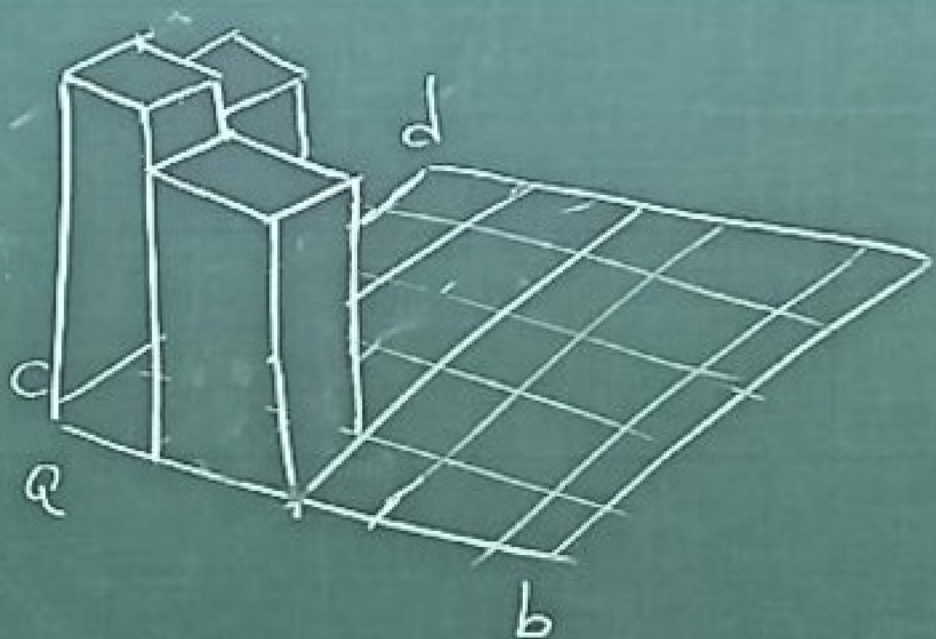


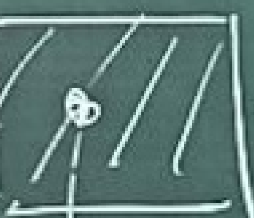




In other words,

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$$


$R_k$:  – kth rectangle

$(x_k, y_k)$ Area $\stackrel{\text{def}}{=} \Delta A_k$

$$P=\left\{\begin{matrix} a=\bar{x}_0<\bar{x}_1<\cdots<\bar{x}_M=b\\ c=\bar{y}_0<\bar{y}_1<\cdots<\bar{y}_N=d\end{matrix}\right\}$$

$$\|P\| = \max_{\text{all } i,\, j}\{\underbrace{\Delta x_i}_{\bar{x}_i-\bar{x}_{i-1}}, \underbrace{\Delta y_j}_{\bar{y}_j-\bar{y}_{j-1}}\}$$

$(x_k, y_k)\in R_k$

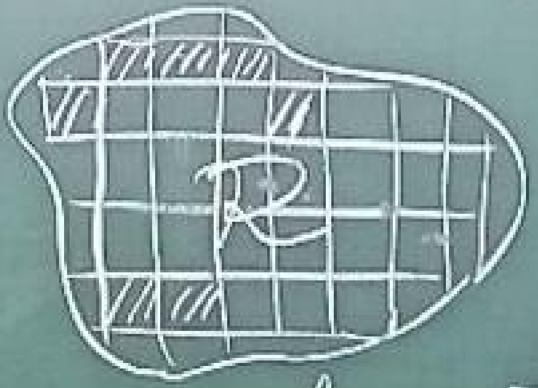
In general, if $f(x,y)$ is continuous on $R$, then $\iint_R f(x,y)\,dA$ exists.
($f$ is integrable on $R$)

<u>Fubini's Theorem.</u>

If $f$ is continuous on $R = [a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

# General Region $R$

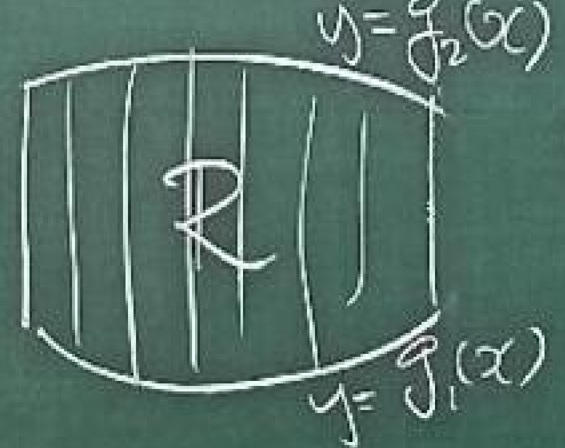


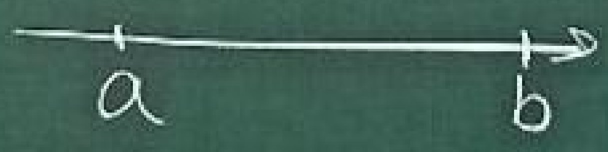

$$\iint_R f(x,y)\,dA$$

$$= \lim_{\substack{\|p\| \to 0 \\ R_k \subset R}} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

Special cases

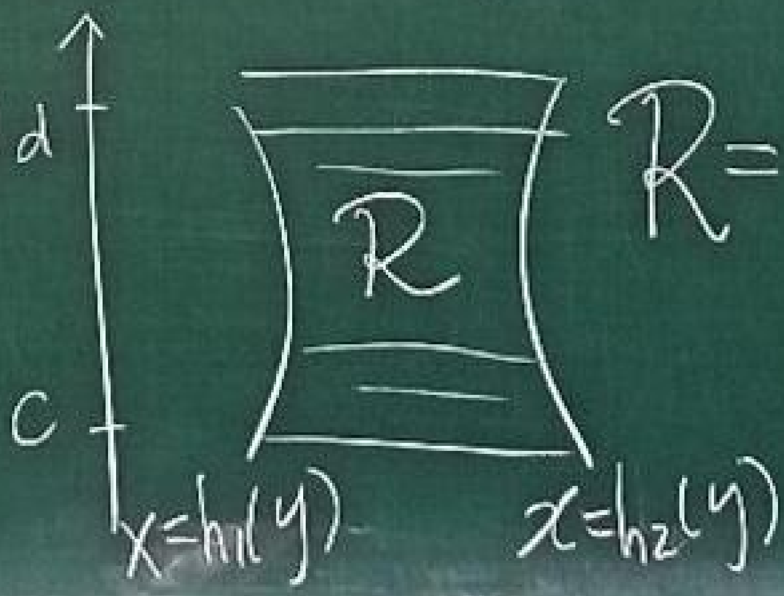
(i)




$$R = \left\{ \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$

(ii)




$$R = \left\{ \begin{array}{l} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right\}$$

# Fubini's Theorem (Strong form)

If $f$ is cont. on $R$ and

(i) $R = \{a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Then $$\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

or

(ii) $R = \{c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

Then $$\iint_R f(x,y)\,dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Eg 1 $R$ = region enclosed by

$x = y$, $x = 1$ and $y = 0$


$$\iint_R \frac{\sin x}{x}\, dA = ?$$

Sol: $R = \{0 \le x \le 1,\ 0 \le y \le x\}$ (i)

$= \{0 \le y \le 1,\ y \le x \le 1\}$ (ii)

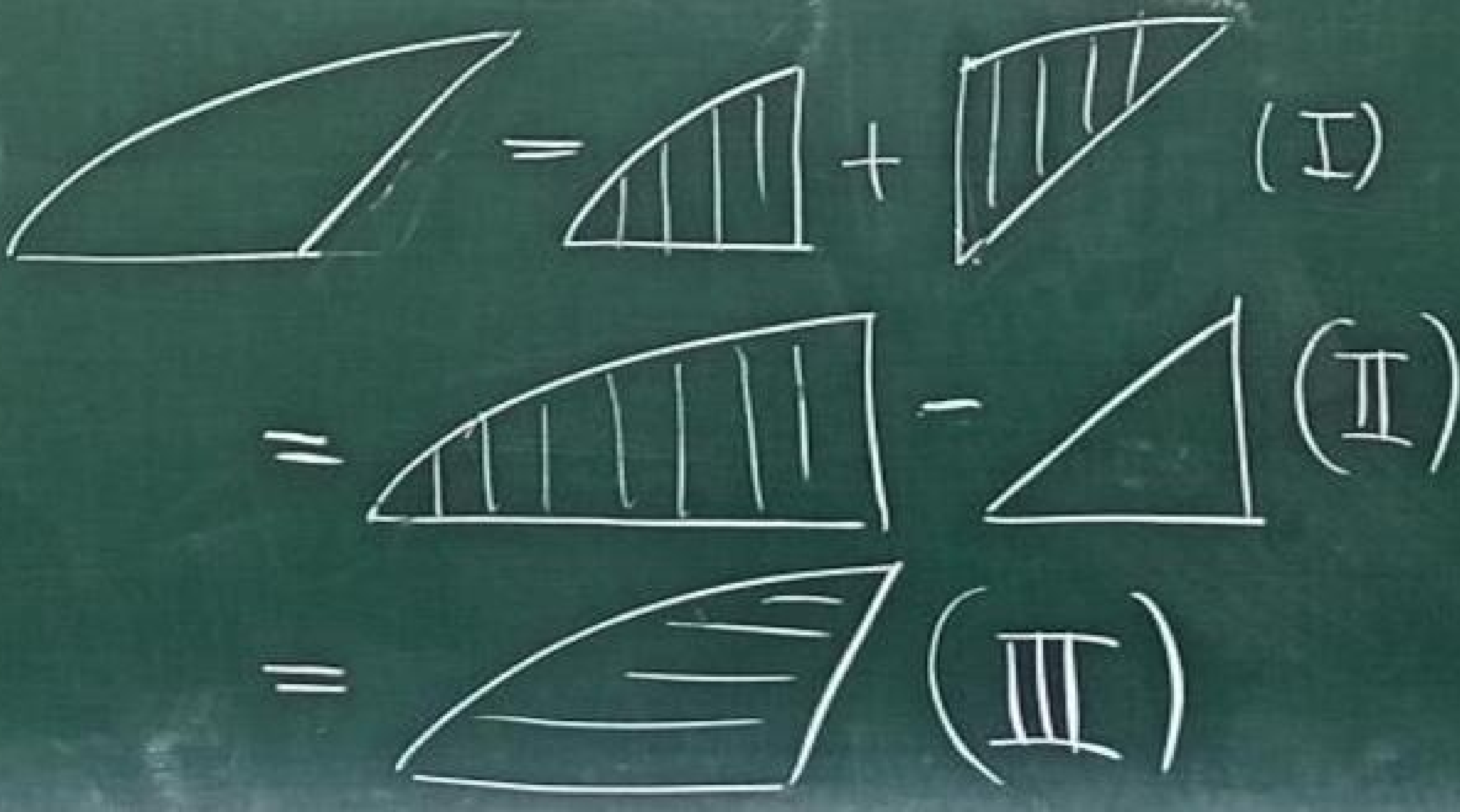
$$\text{(ii)} = \int_{y=0}^{1} \int_{x=y}^{1} \frac{\sin x}{x}\, dx\, dy = ?$$

$$\text{(i)} = \int_{x=0}^{1} \int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \left( \frac{\sin x}{x} \int_{y=0}^{x} 1\, dy \right) dx = -\cos x \Big|_0^1 = 1 - \cos 1$$

Eg2: Find the area of the region enclosed by $\begin{cases} y = 0 \\ y = 4x - 2 \\ x = \dfrac{y^2}{4} \quad (y > 0) \end{cases}$

Sol: $A = \iint_R 1 \, dA$

(I): exercise.

$$(\text{II}) = \int_{x=0}^{1}\int_{y=0}^{2\sqrt{x}} 1\,dy\,dx - \frac{1}{2}\cdot\frac{1}{2}\cdot 2$$

$$= \int_0^1 2\sqrt{x}\,dx - \frac{1}{2} = \frac{5}{6}$$

$$(\text{III}) = \int_{y=0}^{2}\int_{x=\frac{y^2}{4}}^{\frac{y+2}{4}} 1\,dx\,dy$$

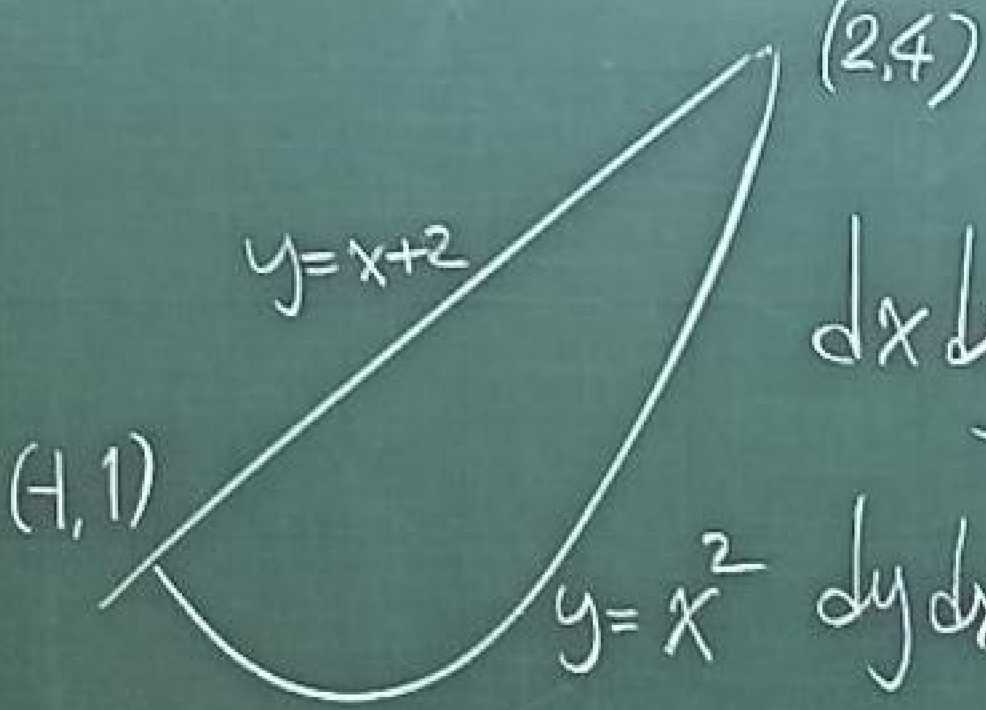
$$= \int_{y=0}^{2}\left(\frac{y+2}{4} - \frac{y^2}{4}\right)dy$$

$$= \int_0^2 -\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2}\,dy$$

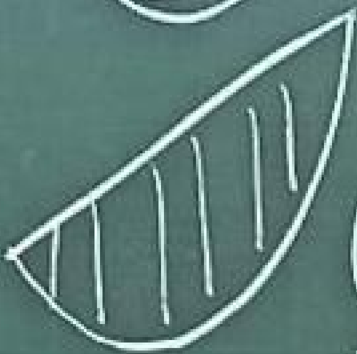
$$= -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2}\Big|_{y=0}^{2}$$

$$= \frac{5}{6}$$

Eg 3. Find area of the region enclosed by $\begin{cases} y = x^2 \\ y = x+2 \end{cases}$




dxdy: 

dydx: 

(Simpler)

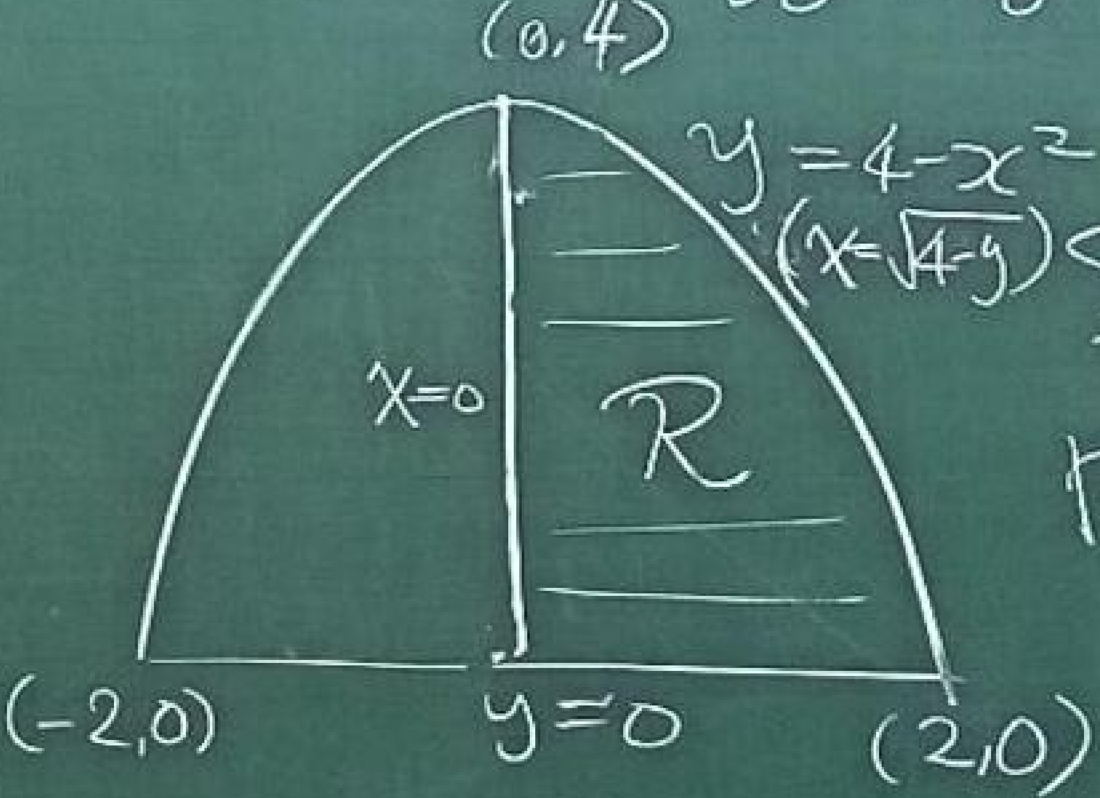
$$A = \int_{x=-1}^{2}\int_{y=x^2}^{x+2} dy\,dx = \int_{-1}^{2} x+2-x^2\,dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2} = \frac{9}{2}$$

Eg4: $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx = ?$

Sol: Try $dx\,dy$ instead.

Note: $\neq \int_0^{4-x^2} \int_0^2 \cdots \, dx \, dy$




Step 1: plot R.

Step 2: Find limits of integration for $dx$

left: $x=0$

Right: $y=4-x^2$, $x=\sqrt{4-y}$

Step 3

$$A = \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} \, dx \, dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \int_{x=0}^{\sqrt{4-y}} x \, dx \, dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \cdot \left( \frac{x^2}{2} \Big|_{0}^{\sqrt{4-y}} \right) dy$$

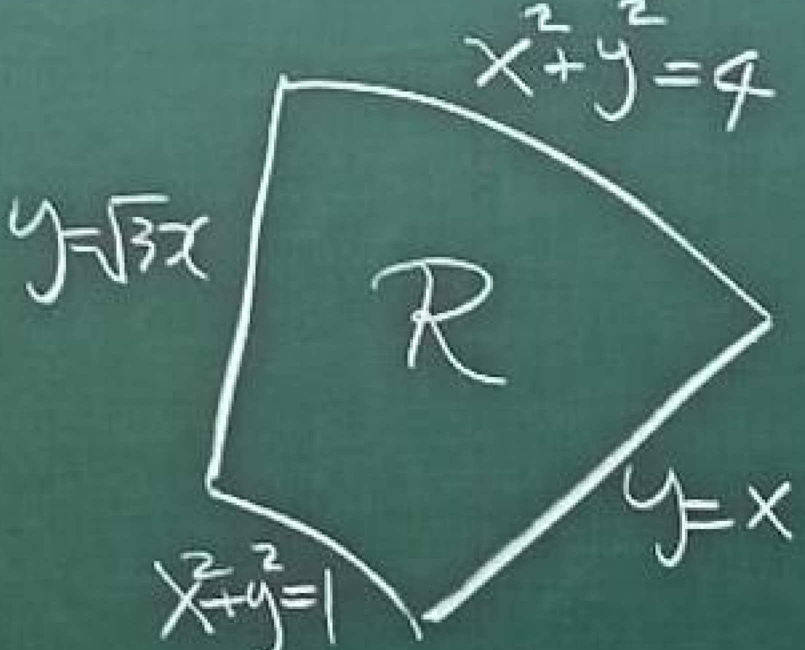
$$= \frac{1}{2} \int_{0}^{4} e^{2y} \, dy$$

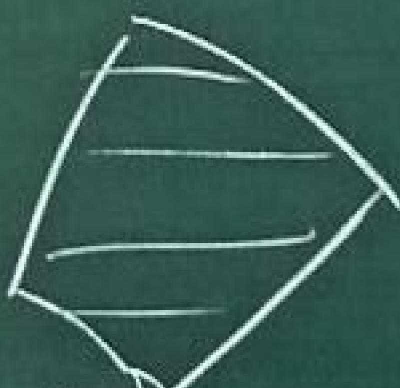
$$= \frac{1}{4} e^{2y} \Big|_{0}^{4} = \frac{e^8 - 1}{4}$$

# Integration in polar coordinate.

**Eg 5** $\iint_R f(x,y)\,dA$

$R$ = region enclosed by $\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2+y^2=1 \\ x^2+y^2=4 \end{cases}$




Both  $(dx\,dy)$

and  $(dy\,dx)$

are complicated.

In this example

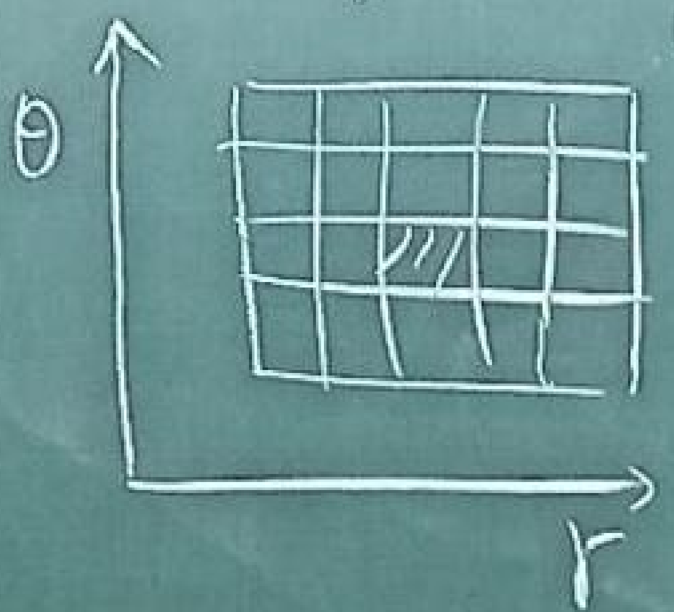
$$R = \left\{ 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3} \right\}$$

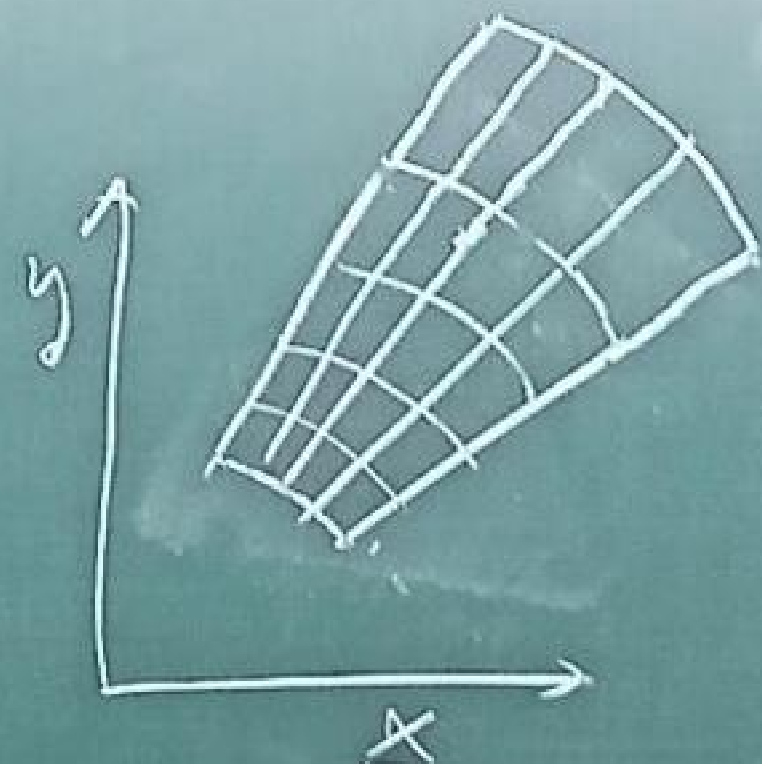
$$I = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} f(\underset{\overset{\shortparallel}{x}}{r\cos\theta}, \underset{\overset{\shortparallel}{y}}{r\sin\theta})\, dA$$

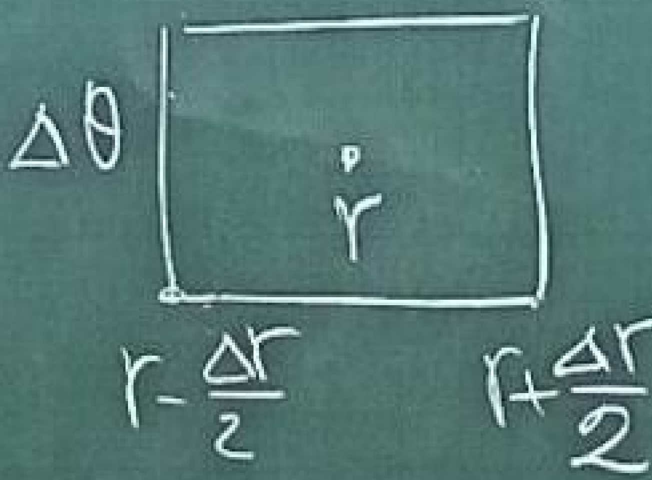
$$dA \overset{?}{=} dr\, d\theta$$

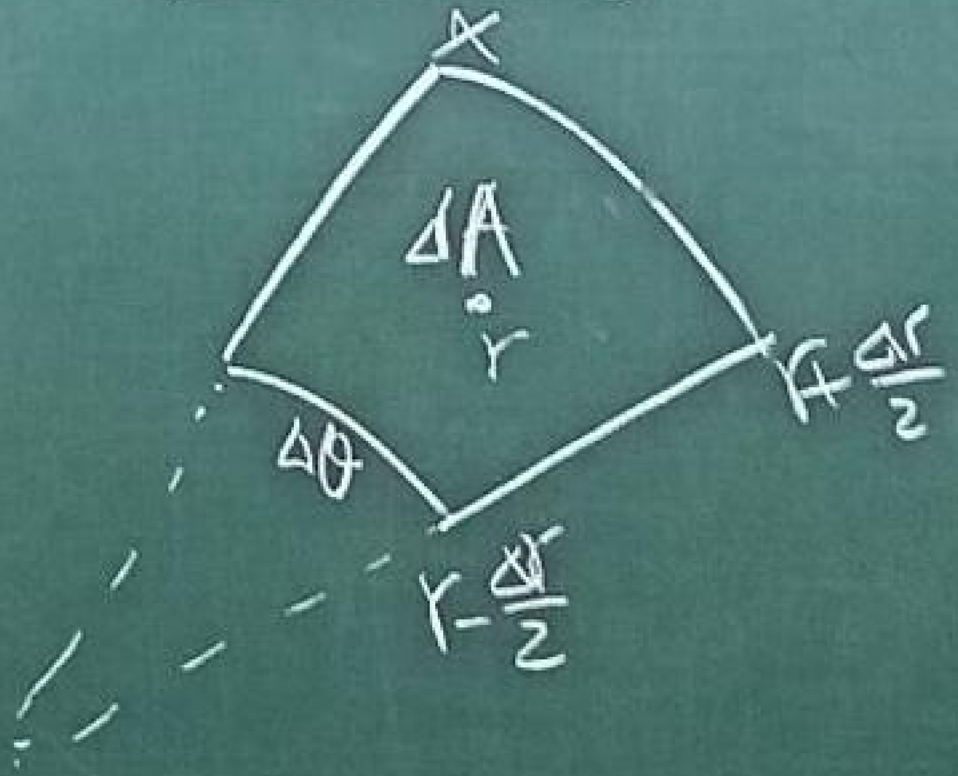
<u>Ans</u>: $dA = r\, dr\, d\theta$

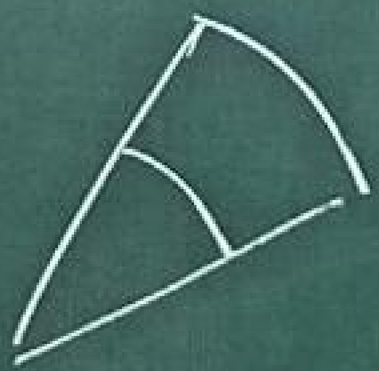
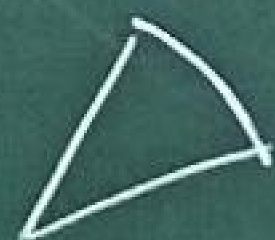
Why?




$\Delta A =$  $-$ 

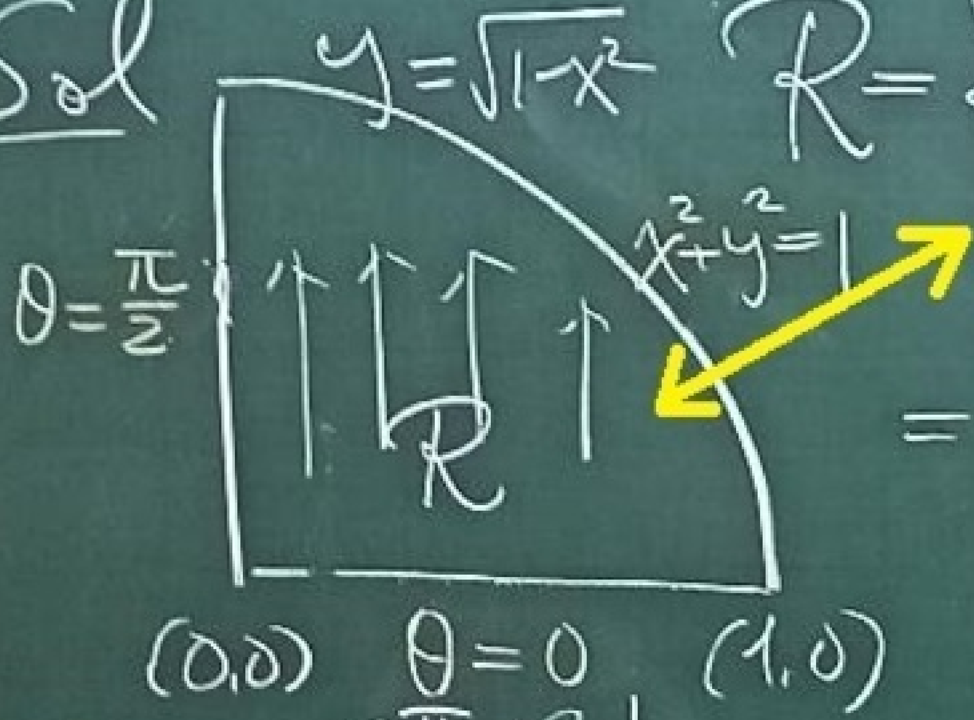
$$= \pi\left(r+\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi} - \pi\left(r-\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi}$$

$$= r\,\Delta r\,\Delta\theta$$

# Integration in Polar Coordinates

**Eg 2** $$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

**Sol**

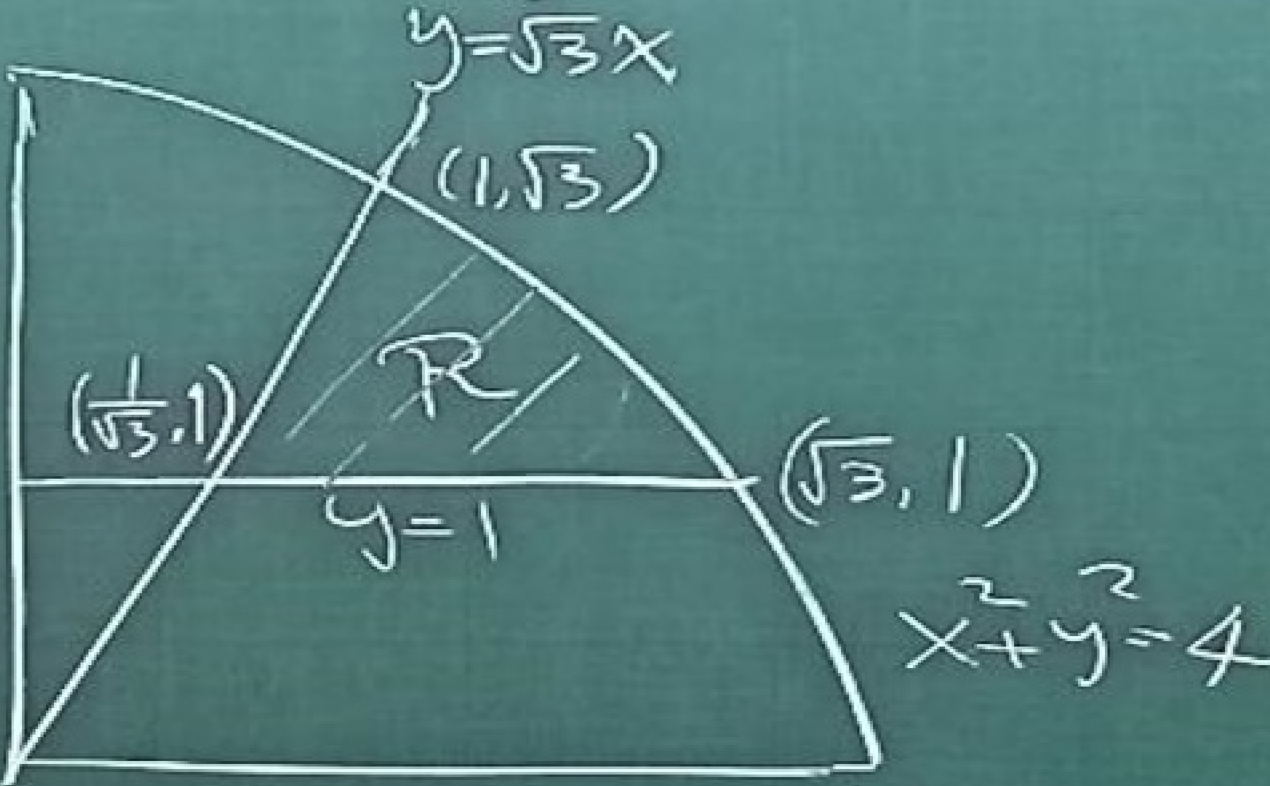



$$R=\left\{\begin{array}{l} 0 \le y \le \sqrt{1-x^2} \\ 0 \le x \le 1 \end{array}\right\} = \left\{\begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{array}\right\}$$

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{1} r^2 \,(\underbrace{r\,dr\,d\theta}_{=\,dA})$$

$$= \left(\int_0^1 r^3\,dr\right)\cdot\left(\int_0^{\frac{\pi}{2}} d\theta\right) = \frac{\pi}{8}$$

Eg 3 Find the area enclosed by $\begin{cases} y=\sqrt{3}x \\ y=1 \\ x^2+y^2=4 \end{cases}$ in 1st quadrant

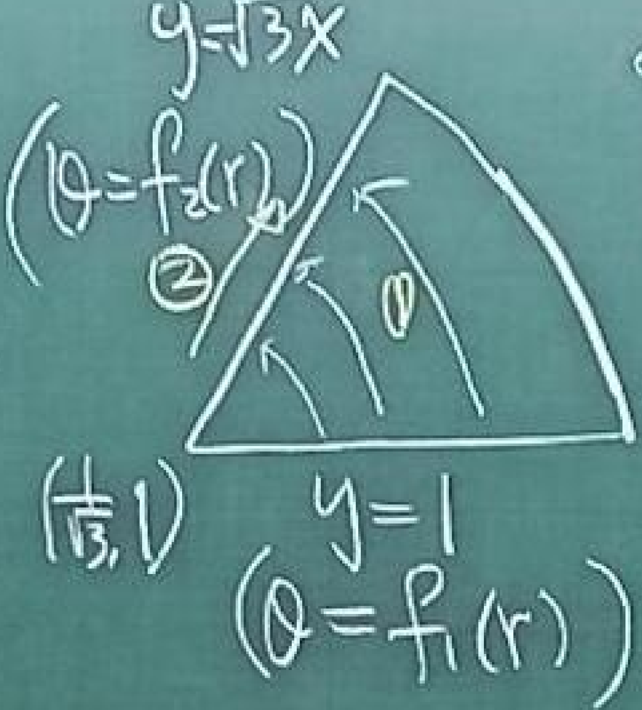



Sol: Method 1:

$A=$  $-$ 

$$=\pi 2^2\cdot\frac{\frac{\pi}{6}}{2\pi}-\frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\cdot 1$$

# Method 2: Polar Coordinate




$$R=\left\{\begin{array}{l} f_1(r)\le\theta\le f_2(r) \\ \underbrace{\frac{2}{\sqrt{3}}}_{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1^2}}\le r\le 2 \end{array}\right\}$$

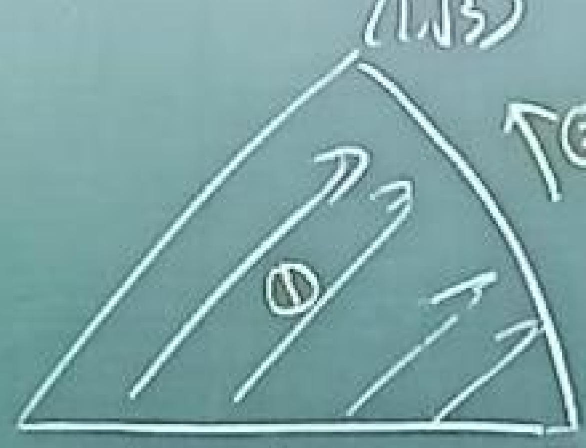
$$y=1 \Leftrightarrow r\sin\theta=1 \Rightarrow \theta=\sin^{-1}\left(\frac{1}{r}\right)$$
$$=f_1(r)$$

$$y=\sqrt{3}x \Leftrightarrow \theta=\frac{\pi}{3}=f_2(r)$$

$$I=\int_{r=\frac{2}{\sqrt{3}}}^{2}\int_{\theta=\sin^{-1}(\frac{1}{r})}^{\frac{\pi}{3}} d\theta\, r\, dr \rightarrow \text{difficult}$$

$$=\int_{\frac{2}{\sqrt{3}}}^{2} r\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{1}{r}\right)\right)dr=?$$

Method 3




$$R=\left\{\begin{array}{l} g(\theta)\le r\le 2 \\ \dfrac{\pi}{6}\le\theta\le\dfrac{\pi}{3} \end{array}\right\}$$

$$\frac{\pi}{6}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right),\qquad \frac{\pi}{3}=\tan^{-1}(\sqrt{3})$$

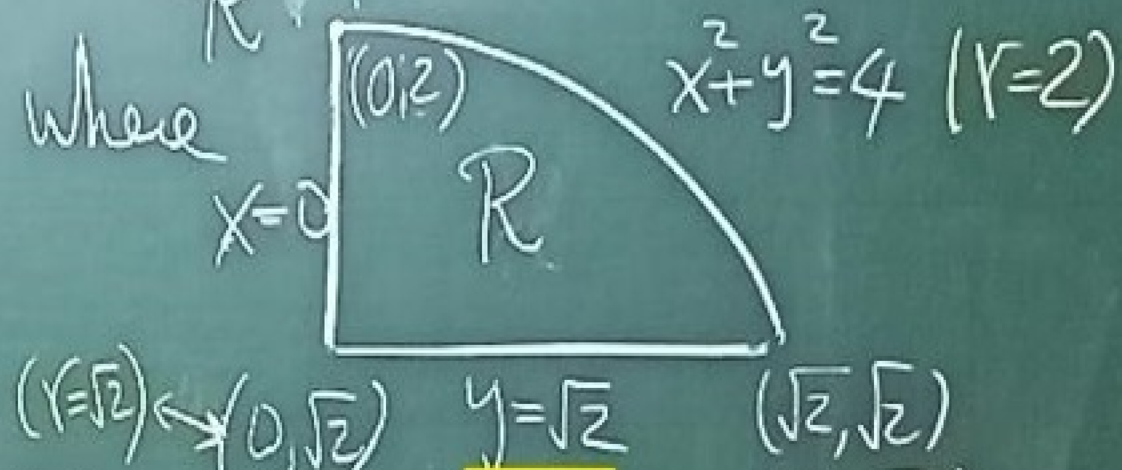
$$y=1 \Leftrightarrow r\sin\theta=1 \Rightarrow r=\frac{1}{\sin\theta}=g(\theta)$$

$$I=\int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{r=\frac{1}{\sin\theta}}^{2} r\,dr\,d\theta$$

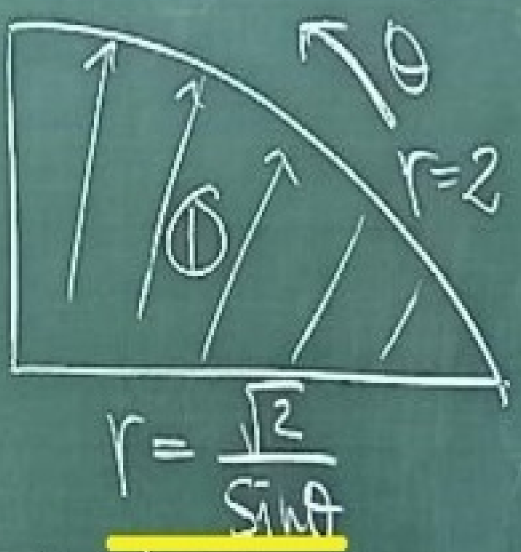
$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(2-\frac{1}{2\sin^2\theta}\right)d\theta$$

$$=\left(2\theta+\frac{\cot\theta}{2}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{3}+\frac{1}{2}\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right)$$

Eg 4: Evaluate $\iint_R f(x,y)\,dA$ in polar coordinate where

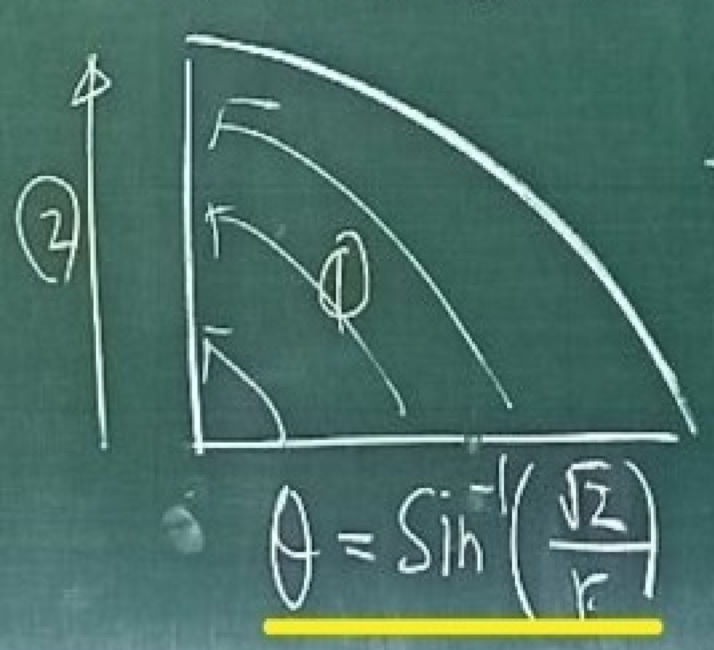



Sol: Method ⓐ $r\,dr\,d\theta$




$$I=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=\frac{2}{\sin\theta}}^{2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

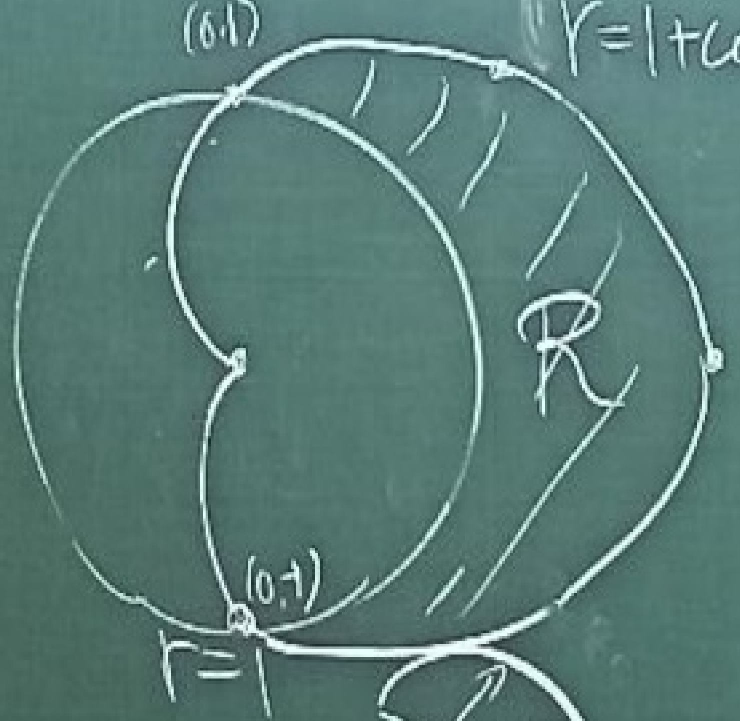
Method ⓑ $d\theta\, r\,dr$




$$I=\int_{r=\sqrt{2}}^{2}\int_{\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)}^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,d\theta\, r\,dr$$

## Eg 5 Evaluate $\iint_R g(r,\theta)\,dA$

where $R$ is the region $\begin{cases} \text{inside } r = 1+\cos\theta \\ \text{outside } r = 1 \end{cases}$




$r = 1+\cos\theta \Rightarrow \cos\theta = r-1$

$$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0, \frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [\frac{-\pi}{2}, 0] \end{cases}$$

(a) $r\,dr\,d\theta$ :
(usually preferred)


$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

(b) $d\theta\,dr$

(usually not good)


$$I = \int_{r=1}^{2} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r,\theta)\, d\theta\, r\,dr$$

## Remark

$r\,dr\,d\theta$ is usually a better choice than $d\theta\, r\,dr$ since most curves in polar coordinates appear naturally in the form $r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r\,dr$