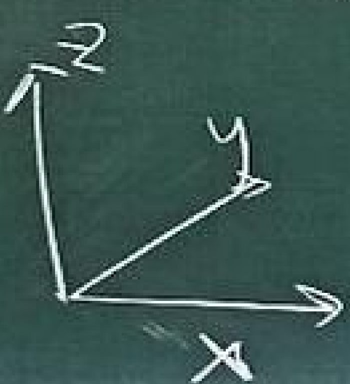
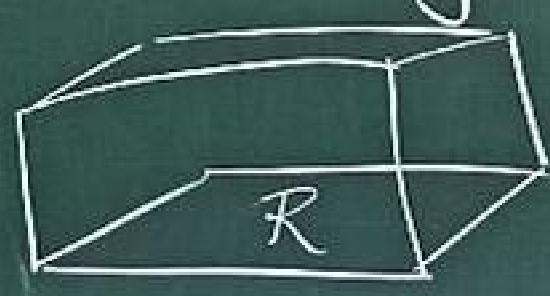
Double Integral

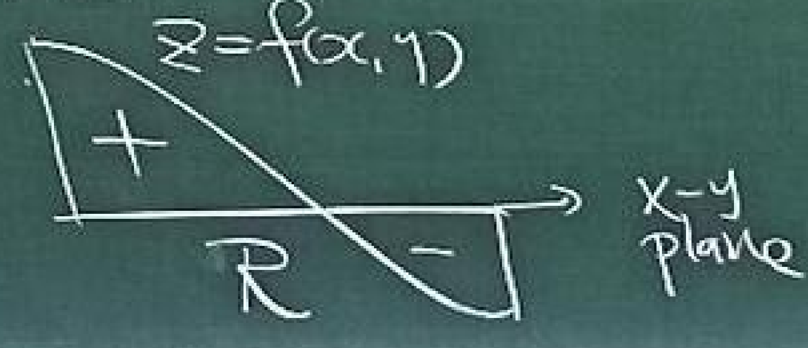
Def: Let $R = [a,b] \times [c,d]$
$= \{ a \le x \le b,\ c \le y \le d \}$

$$\iint_R f(x,y)\, dA$$

= Signed Volume between
$z = f(x,y)$ graph and x-y plane
over the region R.

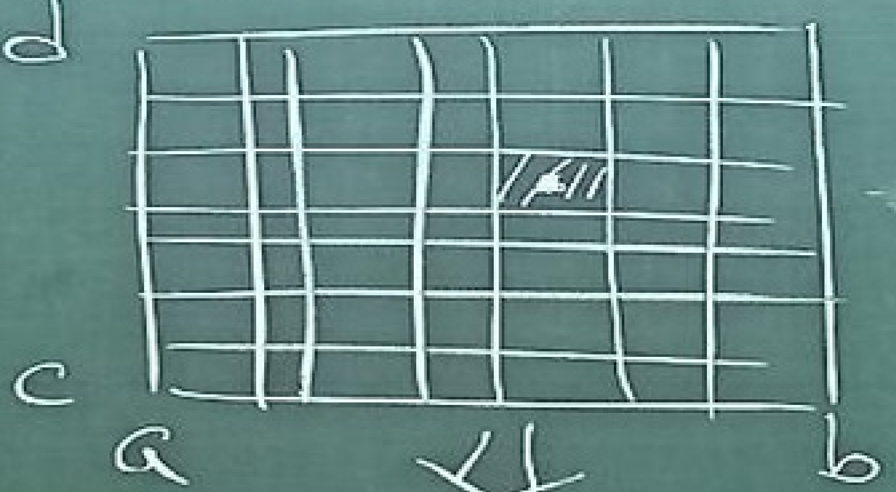


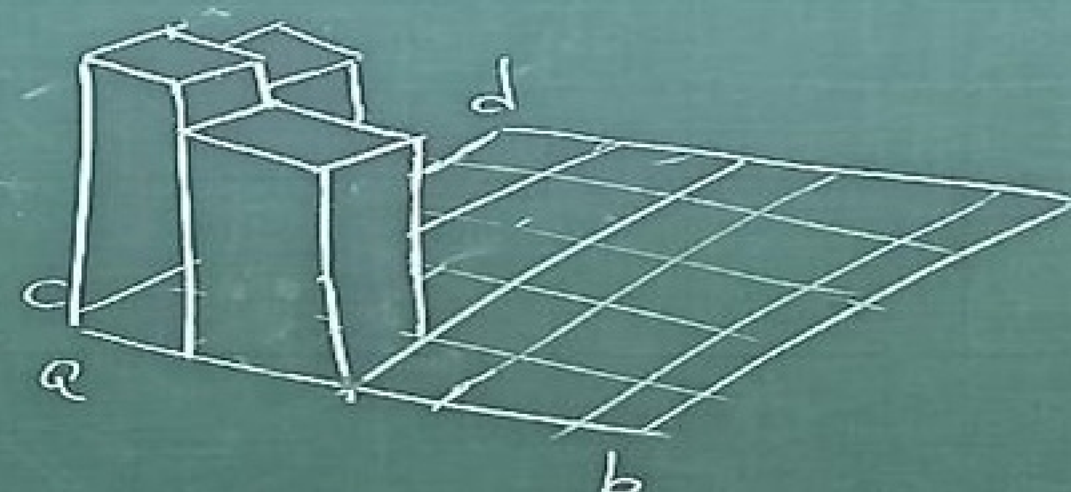




In other words,

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$





R_k: $\boxed{}$ — kth rectangle

(x_k, y_k) Area $\stackrel{\text{def}}{=} \Delta A_k$

$$P = \left\{ \begin{array}{l} a = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_M = b \\ c = \bar{y}_0 < \bar{y}_1 < \cdots < \bar{y}_N = d \end{array} \right\}$$

$$\|P\| = \max_{\text{all } i,\, j} \{ \Delta x_i, \Delta y_j \}$$

$\Delta x_i = \bar{x}_i - \bar{x}_{i-1}$, $\Delta y_j = \bar{y}_j - \bar{y}_{j-1}$

$(x_k, y_k) \in R_k$

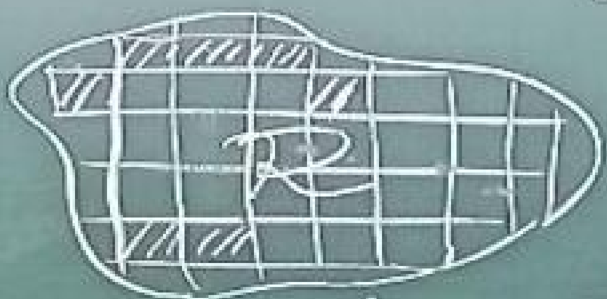
In general, if $f(x,y)$ is continuous on R, then $\iint_R f(x,y)\,dA$ exists.
(f is integrable on R)

<u>Fubini's Theorem.</u>

If f is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

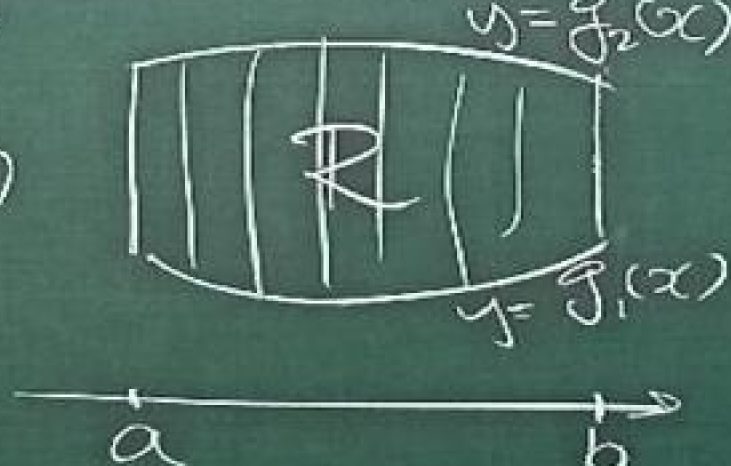
General Region R



$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{\substack{k=1 \\ R_k \subset R}}^{n} f(x_k, y_k)\,\Delta A_k$$

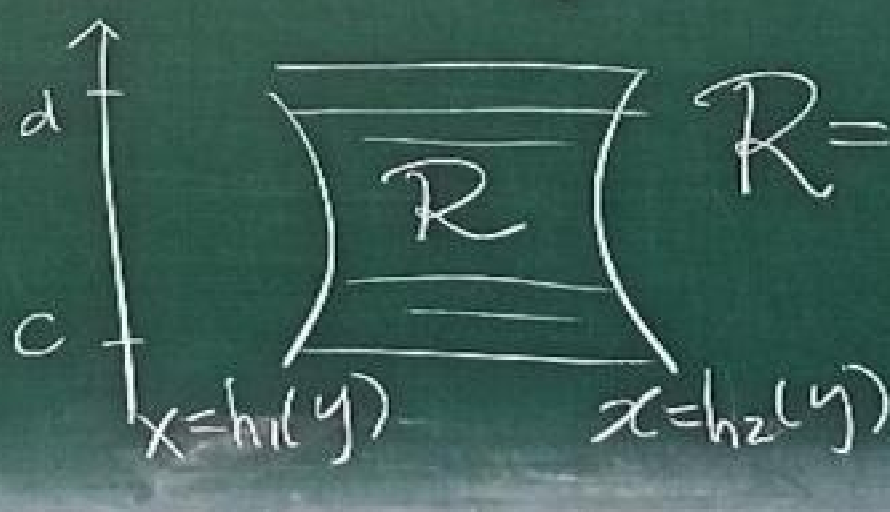
Special cases

(i)



$$R = \left\{ \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$

(ii)



$$R = \left\{ \begin{array}{l} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right\}$$

Fubini's Theorem (Strong form)

If f is cont. on R and

(i) $R = \{a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

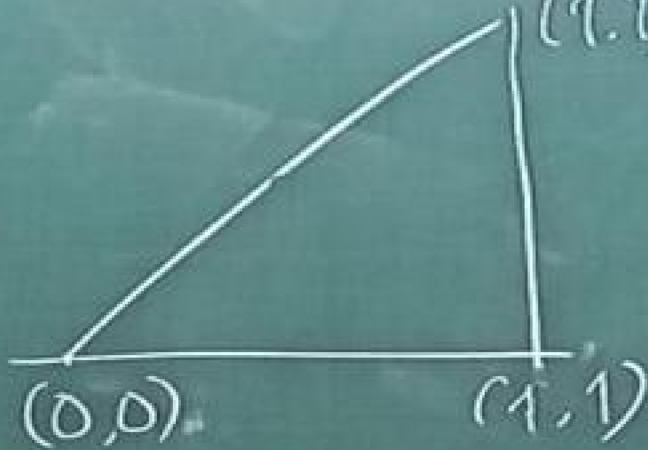
Then $$\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

or

(ii) $R = \{c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

Then $$\iint_R f(x,y)\,dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Eg 1 R = region enclosed by $x=y$, $x=1$ and $y=0$



$$\iint_R \frac{\sin x}{x}\, dA = ?$$

Sol: $R = \{0 \le x \le 1,\ 0 \le y \le x\}$ (i) 

$= \{0 \le y \le 1,\ y \le x \le 1\}$ (ii) 

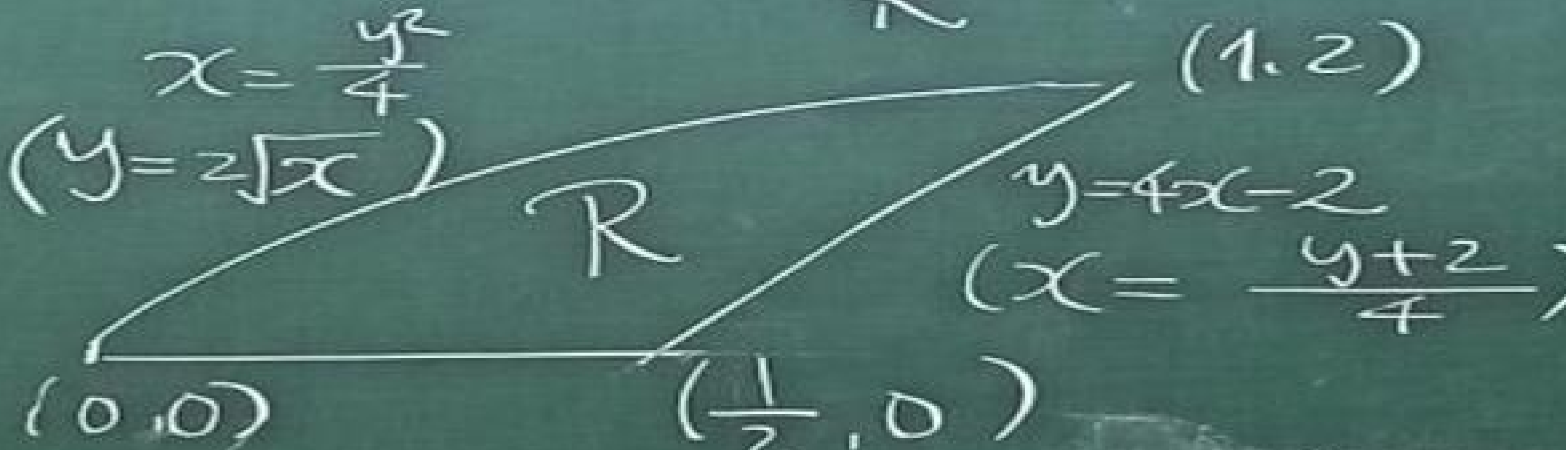
$$(ii) = \int_{y=0}^{1} \int_{x=y}^{1} \frac{\sin x}{x}\, dx\, dy = ?$$

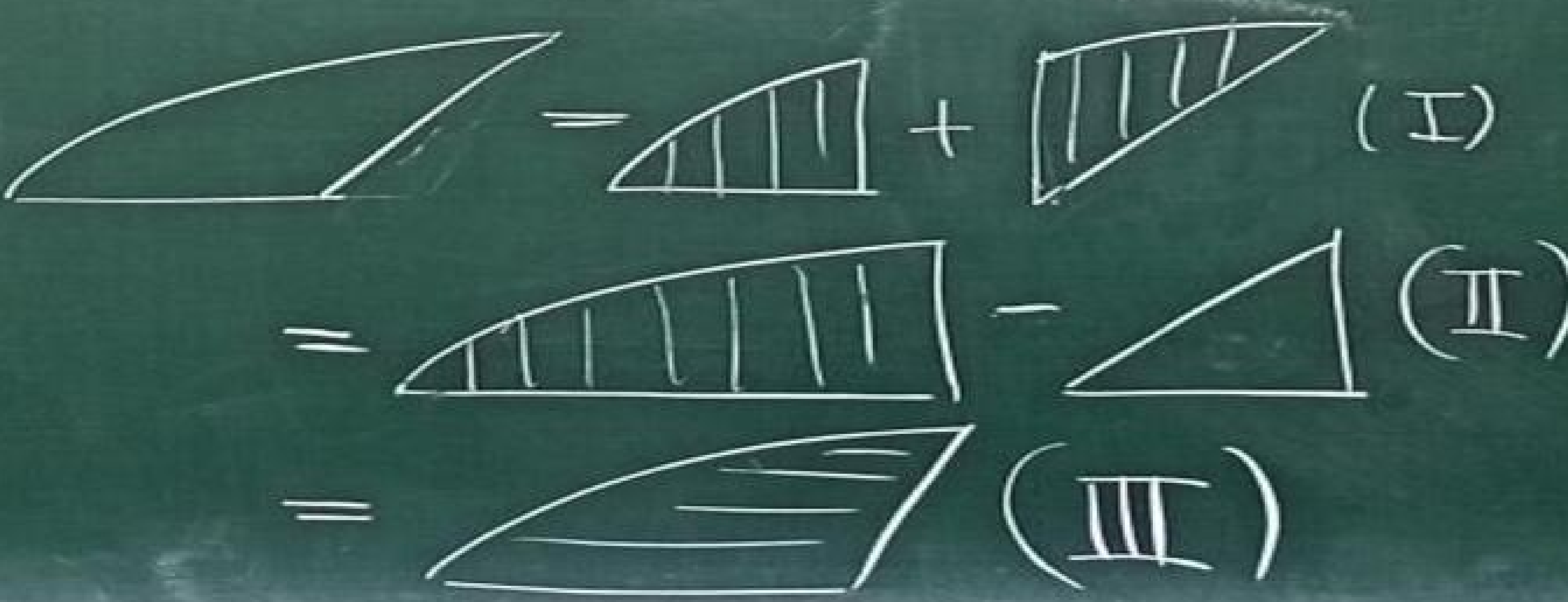
$$(i) = \int_{x=0}^{1} \int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \left(\frac{\sin x}{x} \int_{y=0}^{x} 1\, dy \right) dx = -\cos x \Big|_0^1 = 1 - \cos 1$$

Eg2: Find the area of the region enclosed by $\begin{cases} y = 0 \\ y = 4x - 2 \\ x = \dfrac{y^2}{4} \\ (y > 0) \end{cases}$

Sol: $A = \iint_R 1 \, dA$





(I): exercise.

$$(\mathrm{II}) = \int_{x=0}^{1} \int_{y=0}^{2\sqrt{x}} 1 \, dy \, dx - \frac{1}{2} \cdot \frac{1}{2} \cdot 2$$

$$= \int_{0}^{1} 2\sqrt{x} \, dx - \frac{1}{2} = \frac{5}{6}$$

$$(\mathrm{III}) = \int_{y=0}^{2} \int_{x=\frac{y^2}{4}}^{\frac{y+2}{4}} 1 \, dx \, dy$$

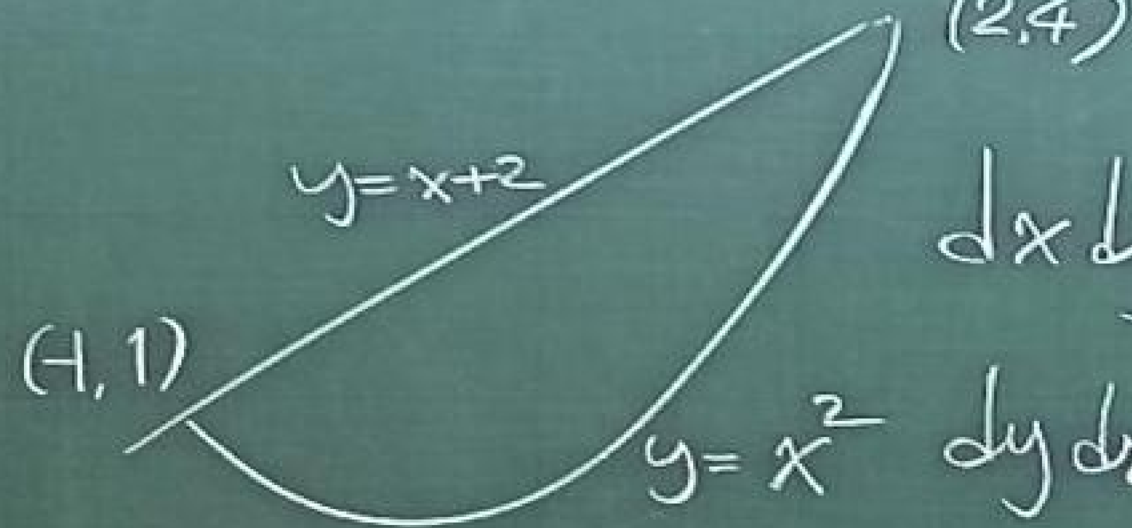
$$= \int_{y=0}^{2} \left(\frac{y+2}{4} - \frac{y^2}{4} \right) dy$$

$$= \int_{0}^{2} -\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2} \, dy$$

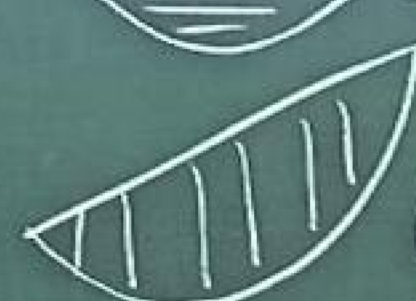
$$= -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2} \Big|_{y=0}^{2}$$

$$= \frac{5}{6}$$

Eg 3. Find area of the region enclosed by $\begin{cases} y = x^2 \\ y = x+2 \end{cases}$



dx dy: 

dy dx:  (Simpler)

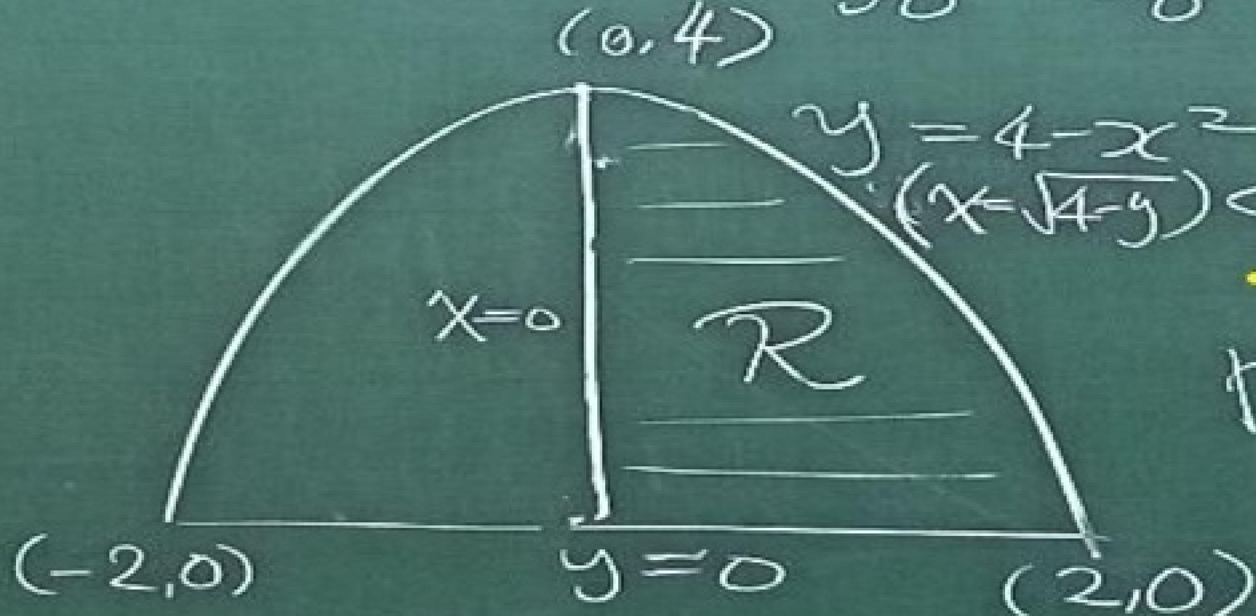
$$A = \int_{x=-1}^{2}\int_{y=x^2}^{x+2} dy\,dx = \int_{-1}^{2} x+2-x^2\,dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2} = \frac{9}{2}$$

Eg4: $\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y}\, dy\, dx = ?$

Sol: Try $dx\,dy$ instead.

Note: $\neq \int_0^{4-x^2} \int_0^2 \cdots \, dx\, dy$



Step 1: plot R.

Step 2: Find limits of integration for dx

left: $x = 0$

Right: $y = 4 - x^2$, $x = \sqrt{4-y}$

Step 3

$$A = \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} \, dx \, dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \int_{x=0}^{\sqrt{4-y}} x \, dx \, dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \cdot \left(\frac{x^2}{2} \Big|_0^{\sqrt{4-y}} \right) dy$$

$$= \frac{1}{2} \int_0^4 e^{2y} \, dy$$

$$= \frac{1}{4} e^{2y} \Big|_0^4 = \frac{e^8 - 1}{4}$$