Partial Derivative with Constraint

Eg1 Find $\frac{\partial w}{\partial x}$ if $\begin{cases} w = x^2+y^2+z^2 \\ z = x^2+y^2 \end{cases}$

Sol total 4 variables + 2 eqns.

3 variable + 1 eqn

$$x+y-z=5$$

$$\Rightarrow x = f(y,z)$$ → independent variables

(or $y = g(z,x)$, $z = h(x,y)$)

3 Variables + 2 eqns

$$\begin{cases} x+y-z=5 \\ x-y+2z=6 \end{cases}$$

$$\Rightarrow \begin{cases} x=f_1(z) \\ y=f_2(z) \end{cases} \text{ or } \begin{cases} y=g_1(x) \\ z=g_2(x) \end{cases} \text{ or } \begin{cases} z=h_1(y) \\ x=h_2(y) \end{cases}$$

4 variables 2 eqns

$\frac{\partial w}{\partial x} \rightarrow$	dependent	independent
	w	x
	(w, y)	(x, z)
	or (w, z)	(x, y)

2 possibilities

$$\begin{cases} w = f_1(x,z) \\ y = g(x,z) \end{cases} \text{ or } \begin{cases} w = f_2(x,y) \\ z = h(x,y) \end{cases}$$

$$\frac{\partial w}{\partial x} = \partial_x f_1(x,z) \quad \text{or} \quad \frac{\partial w}{\partial x} = \partial_x f_2(x,y)$$

They are different! Need to specify $\left(\frac{\partial w}{\partial x}\right)_y$ or $\left(\frac{\partial w}{\partial x}\right)_z$

$\left(\frac{\partial w}{\partial x}\right)_y$: $\begin{cases} w = f_2(x,y) = x^2+y^2+(x^2+y^2)^2 \\ z = h(x,y) = x^2+y^2 \end{cases}$

$\left(\frac{\partial w}{\partial x}\right)_z$: $\begin{cases} w = f_1(x,z) = z + z^2 \\ y = g(x,z) = \pm\sqrt{z-x^2} \end{cases}$

Eg2. Find $\left(\frac{\partial w}{\partial x}\right)_y$ at $(x,y,z) = (2,-1,1)$

if $\begin{cases} w = x^2+y^2+z^2 \\ z^3 - xy + yz + y^3 = 1 \end{cases}$

Ans $\left(\frac{\partial w}{\partial x}\right)_y \Rightarrow \begin{cases} w = w(x,y) \\ z = z(x,y) \end{cases}$

$\Rightarrow w_x = 2x + 2z z_x$ (1)

$3z^2 z_x - y + y z_x = 0$ (2)

(2) $\Rightarrow \left(\frac{\partial z}{\partial x}\right)_y (2,-1,1) = \frac{y}{3z^2+y} = \frac{-1}{2}$

(1) $\Rightarrow \left(\frac{\partial w}{\partial x}\right)_y (2,-1,1) = 2\cdot 2 + 2\cdot 1\cdot\frac{-1}{2} = 3$

Eg3 Find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $\begin{cases} w = x^2 + y - z + \sin t \\ x + y = t \end{cases}$

Sol: 5 variables + 2 eqns

$\Rightarrow$ 2 dependent + 3 independent

$\left(\frac{\partial w}{\partial x}\right)_{y,z}$ = $\begin{matrix} w = w(x,y,z) \\ t = t(x,y,z) \end{matrix}$

$\Rightarrow w_x = 2x + 0 - 0 + \cos(x+y)$ #

Eg4: Find $\left(\frac{\partial z}{\partial x}\right)_y$ if $\begin{cases} f(x,y,z,w) = 0 \\ g(x,y,z,w) = 0 \end{cases}$

Ans: $z = z(x,y)$, $w = w(x,y)$

$\left(\frac{\partial}{\partial x}\right)_y \Rightarrow \begin{cases} f_x + f_z z_x + f_w w_x = 0 \\ g_x + g_z z_x + g_w w_x = 0 \end{cases}$

$\Rightarrow$ 2 unknowns (z_x, w_x) + 2 linear eqns

$\Rightarrow$ Solve for z_x and w_x

Eg4. Find $\left(\frac{\partial(PV)}{\partial T}\right)_{n,V}$

(*) if $PV = nRT$, R = absolute constant

Sol: Closed system: n = fixed

4 variables + 1 eqn (*)
(P, V, n, T)

$\left(\frac{\partial(PV)}{\partial T}\right)_{n,V} \Rightarrow P = P(n, V, T)$

$\frac{\partial(P(n,V,T)\cdot V)}{\partial T} = P_T \cdot V$

$(*) \Rightarrow P_T V = nR$ #

Eg5: Find $\left(\frac{\partial(PV)}{\partial T}\right)_n$ if $PV = nRT$

Sol $\left(\frac{\partial(PV)}{\partial T}\right)_n \Rightarrow P = f(n,T)$, $V = g(n,T)$

Explicit form of f, g are not known

except $f(T,n)\,g(T,n) = nRT$

$$\Rightarrow \left(\frac{\partial(PV)}{\partial T}\right)_n = \frac{\partial(f(T,n)\,g(T,n))}{\partial T}$$

$$= \frac{\partial}{\partial T}(nRT) = nR \quad \text{(HW)}$$