Constrained Variables

Partial Derivative with ~~Constraint~~

Eg1. Find $\frac{\partial w}{\partial x}$ if $\begin{cases} w = x^2 + y^2 + z^2 \\ z = x^2 + y^2 \end{cases}$

Sol total 4 variables + 2 eqns.

<u>3 variables + 1 eqn</u>

$$x + y - z = 5$$

$$\Rightarrow x = f(y, z)$$ indepedent variables

$$(\text{or } y = g(z, x),\ z = h(x, y))$$

<u>3 Variables + 2 eqns</u>

$$\begin{cases} x + y - z = 5 \\ x - y + 2z = 6 \end{cases}$$

$$\Rightarrow \begin{cases} x = f_1(z) \\ y = f_2(z) \end{cases} \text{ or } \begin{cases} y = g_1(x) \\ z = g_2(x) \end{cases} \text{ or } \begin{cases} z = h_1(y) \\ x = h_2(y) \end{cases}$$

4 variables 2 eqns

dependent independent

$\frac{\partial w}{\partial x} \rightarrow$ w x

(w, y) (x, z)

or (w, z) (x, y)

2 possibilities

(A): $\begin{cases} w = f_1(x, z) \\ y = g(x, z) \end{cases}$ or (B): $\begin{cases} w = f_2(x, y) \\ z = h(x, y) \end{cases}$

$\frac{\partial w}{\partial x} = \partial_x f_1(x, z)$ or $\frac{\partial w}{\partial x} = \partial_x f_2(x, y)$

They are different! Need to specify $\overset{(B)}{\left(\frac{\partial w}{\partial x}\right)_y}$ or $\overset{(A)}{\left(\frac{\partial w}{\partial x}\right)_z}$

($\frac{\partial w}{\partial x}$ alone, is not a valid question)

$$\left(\frac{\partial w}{\partial x}\right)_y : \begin{cases} w = f_2(x,y) = x^2+y^2+(x^2+y^2)^2 \\ z = h(x,y) = x^2+y^2 \end{cases}$$

$$\left(\frac{\partial w}{\partial x}\right)_z : \begin{cases} w = f_1(x,z) = z+z^2 \\ y = g(x,z) = \pm\sqrt{z-x^2} \end{cases}$$

Eg2. Find $\left(\frac{\partial w}{\partial x}\right)_y$ at $(x,y,z) = (2,-1,1)$

if $\begin{cases} w = x^2+y^2+z^2 \\ z^3 - xy + yz + y^3 = 1 \end{cases}$

<u>Ans</u> $\left(\frac{\partial w}{\partial x}\right)_y \Rightarrow \begin{cases} w = w(x,y) \\ z = z(x,y) \end{cases}$

$$\Rightarrow w_x = 2x + 2zz_x \quad (1)$$

$$3z^2 z_x - y + yz_x = 0 \quad (2)$$

$$(2) \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y (2,-1,1) = \frac{y}{3z^2+y} = \frac{-1}{2}$$

$$(1) \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y (2,-1,1) = 2\cdot 2 + 2\cdot 1\cdot \frac{-1}{2} = \underline{\underline{3}}$$

Eg3 Find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $\begin{cases} w = x^2 + y - z + \sin t \\ x + y = t \end{cases}$

Sol: 5 variables + 2 eqns

$\Rightarrow$ 2 dependent + 3 independent

$\left(\frac{\partial w}{\partial x}\right)_{y,z} = \quad \begin{matrix} w = w(x, y, z) \\ t = t(x, y, z) \end{matrix}$

$\Rightarrow w_x = 2x + 0 - 0 + \cos(x+y)$ #

General Procedure: Find $\left(\frac{\partial z}{\partial x}\right)_y$ if $\begin{cases} f(x, y, z, w) = 0 \\ g(x, y, z, w) = 0 \end{cases}$

Ans: $z = z(x, y)$, $w = w(x, y)$

$\left(\frac{\partial}{\partial x}\right)_y \Rightarrow \begin{cases} f_x + f_z z_x + f_w w_x = 0 \\ g_x + g_z z_x + g_w w_x = 0 \end{cases}$

$\Rightarrow$ 2 unknowns (z_x, w_x) + 2 linear eqns

$\Rightarrow$ Solve for z_x and w_x

Eg4. Find $\left(\frac{\partial(PV)}{\partial T}\right)_{n,V}$

(*) if $PV = nRT$, R = absolute constant

<u>Sol</u>: closed system: n = fixed

4 variables + 1 eqn (*)
(P, V, n, T)

$$\left(\frac{\partial(PV)}{\partial T}\right)_{n,V} \Rightarrow P = P(n, V, T)$$

$$\frac{\partial(P(n,V,T)\cdot V)}{\partial T} = P_T \cdot V$$

$$(*) \Rightarrow P_T V = nR \ \#$$

Eg5: Find $\left(\frac{\partial(PV)}{\partial T}\right)_n$ if $PV = nRT$

Sol $\left(\frac{\partial(PV)}{\partial T}\right)_n \Rightarrow \begin{array}{l} P = f(n,T) \\ V = g(n,T) \end{array}$

Explicit form of f, g are not known

except $f(T,n)\, g(T,n) = nRT$

$$\Rightarrow \left(\frac{\partial(PV)}{\partial T}\right)_n = \frac{\partial(f(n,T)\, g(n,T))}{\partial T}$$

$$= \frac{\partial}{\partial T}(nRT) = nR \quad (HW)$$

Eg 6. Find local extremes

of $w = x^2 + y^2$ on $\{(x,y) \mid (x-\frac{1}{2})^2 + \frac{y^2}{4} = 1\}$

<u>Sol</u> 3 unknowns (w, x, y), 2 eqns.

$\Rightarrow$ Either $\begin{cases} w = w(y) \\ x = x(y) \end{cases}$ or $\begin{cases} w = w(x) \\ y = y(x) \end{cases}$ (NG)

Find y_0 and $x_0 \stackrel{def}{=} x(y_0)$ such that $w'(y_0) = 0$

$(g=0)$ $(x_0 - \frac{1}{2})^2 + \frac{y_0^2}{4} = 1 \quad \cdots (1)$

$(w'(y_0)=)$ $2x_0 x_0' + 2y_0 = 0 \quad \cdots (2')$

Need 1 more eq for x_0'

$(\frac{d}{dy} g\big|_{y_0} = 0)$ $2(x_0 - \frac{1}{2})x_0' + \frac{y_0}{2} = 0 \cdots (3')$

<u>Compare</u>: Method of Lagrangian multiplier

$(x_0 - \frac{1}{2})^2 + \frac{y_0^2}{4} = 1$

$2x_0 = 2\lambda(x_0 - \frac{1}{2}) \cdots (2)$

$2y_0 = \lambda \cdot \frac{y_0}{2} \quad \cdots (3)$

Rm: $(2'), (3') \Rightarrow \dfrac{2x_0}{2(x_0 - \frac{1}{2})} = \dfrac{2y_0}{\frac{y_0}{2}}$ (call it $= \lambda$)

$\Rightarrow (2), (3)$

Solve (x_0, y_0, x_0')
from (1), (2') and (3')

$$\xrightarrow{(2')-4(3')} \quad -6x_0x_0' = -4x_0' \cdots (4)$$

$$\xrightarrow{(2')-(3')} \quad \frac{3}{2}y_0 = -x_0' \cdots (5)$$

$$(4) \Rightarrow \quad x_0' = 0 \cdots (A)$$
$$\text{or } x_0 = \frac{2}{3} \cdots (B)$$

case(A): $(x_0, y_0, x_0') = (\frac{1}{2}, 0, 0)$ ①
(use (1)+(5)) $(\frac{3}{2}, 0, 0)$ ②

case(B): $(x_0, y_0, x_0') = \left(\frac{2}{3}, \frac{\pm\sqrt{35}}{3}, \frac{\mp\sqrt{35}}{2}\right)$ ③ ④

$$w''(y_0) = 2(x_0')^2 + 2x_0\underset{?}{\underline{x_0''}} + 2$$

(Need eqn for x_0''!)

$$(3') \Rightarrow 2x_0'^2 + 2(x_0 - \tfrac{1}{2})x_0'' + \tfrac{1}{2} = 0 \cdots (6)$$

$$\Rightarrow x_0'' = \frac{-x_0'^2 - \frac{1}{4}}{x_0 - \frac{1}{2}} = \underset{①}{\frac{1}{4}}, \underset{②}{\frac{-1}{4}}, \underset{③}{-54}, \underset{④}{-54}$$

$$\Rightarrow \omega_0'' = 2x_0'^2 + 2x_0x_0'' + 2 \overset{(6)}{=} x_0'' + \frac{3}{2}$$

$$= \underset{①}{+} \quad \underset{②}{+} \quad \underset{③}{-} \quad \underset{④}{-}$$

$\therefore$ ① ② = local min
③ ④ = local max