

## Constrained Variables

Partial Derivative with ~~Constraint~~

Ex 1. Find  $\frac{\partial w}{\partial x}$  if  $\begin{cases} w = x^2 + y^2 + z^2 \\ z = x^2 + y^2 \end{cases}$

Sol total 4 variables + 2 eqns.

3 variables + 1 eqn

$$\begin{aligned} x + y - z &= 5 \\ \Rightarrow x &= f(y, z) \quad \begin{array}{l} \nearrow \text{independent} \\ \text{variables} \end{array} \\ & \text{(or } y = g(z, x), z = h(x, y) \text{)} \end{aligned}$$

3 Variables + 2 eqns

$$\begin{aligned} & \begin{cases} x + y - z = 5 \\ x - y + 2z = 6 \end{cases} \\ \Rightarrow & \begin{cases} x = f_1(z) \\ y = f_2(z) \end{cases} \text{ or } \begin{cases} y = g_1(x) \\ z = g_2(x) \end{cases} \text{ or } \begin{cases} z = h_1(y) \\ x = h_2(y) \end{cases} \end{aligned}$$

4 variables 2 eqns

dependent independent  
 $\frac{\partial w}{\partial x} \rightarrow w$   $x$   
 $(w, y)$   $(x, z)$   
or  $(w, z)$   $(x, y)$

2 possibilities

$$(A): \begin{cases} w = f_1(x, z) \\ y = g(x, z) \end{cases} \quad \text{or} \quad (B): \begin{cases} w = f_2(x, y) \\ z = h(x, y) \end{cases}$$

$$\frac{\partial w}{\partial x} = \partial_x f_1(x, z) \quad \text{or} \quad \frac{\partial w}{\partial x} = \partial_x f_2(x, y)$$

They are different! Need to

specify  $\left(\frac{\partial w}{\partial x}\right)_y$  or  $\left(\frac{\partial w}{\partial x}\right)_z$

$\left(\frac{\partial w}{\partial x}\right)$  alone, is not a valid question

$$\left(\frac{\partial \omega}{\partial x}\right)_y: \begin{cases} \omega = f_2(x, y) = x^2 + y^2 + (x^2 + y^2)^2 \\ z = h(x, y) = x^2 + y^2 \end{cases}$$

$$\left(\frac{\partial \omega}{\partial x}\right)_z: \begin{cases} \omega = f_1(x, z) = z + z^2 \\ y = g(x, z) = \pm \sqrt{z - x^2} \end{cases}$$

Ex 2. Find  $\left(\frac{\partial \omega}{\partial x}\right)_y$  at  $(x, y, z) = (2, 1, 1)$

$$\text{if } \begin{cases} \omega = x^2 + y^2 + z^2 \\ z^3 - xy + yz + y^3 = 1 \end{cases}$$

Ans  $\left(\frac{\partial \omega}{\partial x}\right)_y \Rightarrow \begin{cases} \omega = \omega(x, y) \\ z = z(x, y) \end{cases}$

$$\Rightarrow \omega_x = 2x + 2z z_x \quad (1)$$

$$3z^2 z_x - y + yz_x = 0 \quad (2)$$

$$(2) \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y (2, 1, 1) = \frac{y}{3z^2 + y} = \frac{1}{2}$$

$$(1) \Rightarrow \left(\frac{\partial \omega}{\partial x}\right)_y (2, 1, 1) = 2 \cdot 2 + 2 \cdot 1 \cdot \frac{1}{2} = \underline{\underline{3}}$$

Ex 3 Find  $(\frac{\partial w}{\partial x})_{y,z}$  if  $\begin{cases} w = x^2 + y - z + \sin t \\ x + y = t \end{cases}$

Sol: 5 variables + 2 eqns

$\Rightarrow$  2 dependent + 3 independent

$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = \begin{matrix} w = w(x, y, z) \\ t = t(x, y, z) \end{matrix}$$

$$\Rightarrow w_x = 2x + 0 - 0 + \cos(x+y) \neq$$

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**General Procedure:** Find  $(\frac{\partial z}{\partial x})_y$  if  $\begin{cases} f(x, y, z, w) = 0 \\ g(x, y, z, w) = 0 \end{cases}$

Ans:  $z = z(x, y), w = w(x, y)$

$$\left(\frac{\partial}{\partial x}\right)_y \Rightarrow \begin{cases} f_x + f_z z_x + f_w w_x = 0 \\ g_x + g_z z_x + g_w w_x = 0 \end{cases}$$

$\Rightarrow$  2 unknowns ( $z_x, w_x$ ) + 2 linear eqns

$\Rightarrow$  Solve for  $z_x$  and  $w_x$

Eg4. Find  $\left(\frac{\partial(PV)}{\partial T}\right)_{n,V}$

(\*) if  $PV = nRT$ ,  $R = \text{absolute constant}$

Sol. Closed system:  $n = \text{fixed}$

4 variables + 1 eqn (\*)  
(P, V, n, T)

$$\left(\frac{\partial(PV)}{\partial T}\right)_{n,V} \Rightarrow P = P(n, V, T)$$

$$\frac{\partial(P(n, V, T) \cdot V)}{\partial T} = P_T \cdot V$$

$$(*) \Rightarrow P_T V = nR \neq$$

Ex 5: Find  $\left(\frac{\partial(PV)}{\partial T}\right)_n$  if  $PV = nRT$

Sol.  $\left(\frac{\partial(PV)}{\partial T}\right)_n \Rightarrow \begin{matrix} P = f(n, T) \\ V = g(n, T) \end{matrix}$

Explicit form of  $f, g$  are not known

except  $f(T, n)g(T, n) = nRT$

$$\Rightarrow \left(\frac{\partial(PV)}{\partial T}\right)_n = \frac{\partial(f(n, T)g(n, T))}{\partial T}$$

$$= \frac{\partial}{\partial T}(nRT) = nR \quad (\text{HW})$$

Ex 6. Find local extremes

$$\text{of } w = x^2 + y^2 \text{ on } \left\{ (x, y) \mid \left(x - \frac{1}{2}\right)^2 + \frac{y^2}{4} = 1 \right\}$$

Sol 3 unknowns  $(w, x, y)$ , 2 eqns.

$$\Rightarrow \text{Either } \begin{cases} w = w(y) \\ x = x(y) \end{cases} \text{ or } \begin{cases} w = w(x) \\ y = y(x) \end{cases} \text{ (NG)}$$

Find  $y_0$  and  $x_0 \stackrel{\text{def}}{=} x(y_0)$

Such that  $w'(y_0) = 0$

$$(g=0) \quad \left(x_0 - \frac{1}{2}\right)^2 + \frac{y_0^2}{4} = 1 \quad \dots (1)$$

$$(w'(y_0)=) \quad 2x_0 x_0' + 2y_0 = 0 \quad \dots (2')$$

Need 1 more eq for  $x_0'$

$$\left(\frac{d}{dy} g \Big|_{y_0} = 0\right) \quad 2\left(x_0 - \frac{1}{2}\right)x_0' + \frac{y_0}{2} = 0 \quad \dots (3')$$

Compare: Method of Lagrangian multiplier

$$\left(x_0 - \frac{1}{2}\right)^2 + \frac{y_0^2}{4} = 1$$

$$2x_0 = 2\lambda \left(x_0 - \frac{1}{2}\right) \quad \dots (2)$$

$$2y_0 = \lambda \cdot \frac{y_0}{2} \quad \dots (3)$$

$$\text{Rm: } (2'), (3') \Rightarrow \frac{2x_0}{2\left(x_0 - \frac{1}{2}\right)} = \frac{2y_0}{\frac{y_0}{2}} \quad (\text{call it } = \lambda)$$

$$\Rightarrow (2), (3)$$

Solve  $(x_0, y_0, x_0')$   
from (1), (2') and (3')

$$(2') - 4(3') \Rightarrow -6x_0x_0' = -4x_0' \dots (4)$$

$$(2') - (3') \Rightarrow \frac{3}{2}y_0 = -x_0' \dots (5)$$

$$(4) \Rightarrow x_0' = 0 \dots (A)$$

$$\text{or } x_0 = \frac{2}{3} \dots (B)$$

$$\text{case (A): } (x_0, y_0, x_0') = \left(\frac{1}{2}, 0, 0\right) \quad \textcircled{1}$$

$$\text{(use (1) + (5))} \quad \left(\frac{3}{2}, 0, 0\right) \quad \textcircled{2}$$

$$\text{case (B): } (x_0, y_0, x_0') = \left(\frac{2}{3}, \frac{\pm\sqrt{35}}{3}, \frac{\mp\sqrt{35}}{2}\right) \quad \textcircled{3}$$

$\textcircled{4}$

$$W'(y_0) = 2(x_0')^2 + 2x_0x_0'' + 2$$

(Need eqn for  $x_0''$ !) ?

$$(3') \Rightarrow 2x_0'^2 + 2\left(x_0 - \frac{1}{2}\right)x_0'' + \frac{1}{2} = 0 \dots (6)$$

$$\Rightarrow x_0'' = \frac{-x_0'^2 - \frac{1}{4}}{x_0 - \frac{1}{2}} = \frac{1}{4}, \frac{1}{4}, -54, -54$$

$\textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \textcircled{4}$

$$\Rightarrow W_0'' = 2x_0'^2 + 2x_0x_0'' + 2 \stackrel{(6)}{=} x_0'' + \frac{3}{2}$$

$\therefore \textcircled{1} \textcircled{2} = \text{local min}$

$\textcircled{3} \textcircled{4} = \text{local max}$

$= + + - -$   
 $\textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \textcircled{4}$