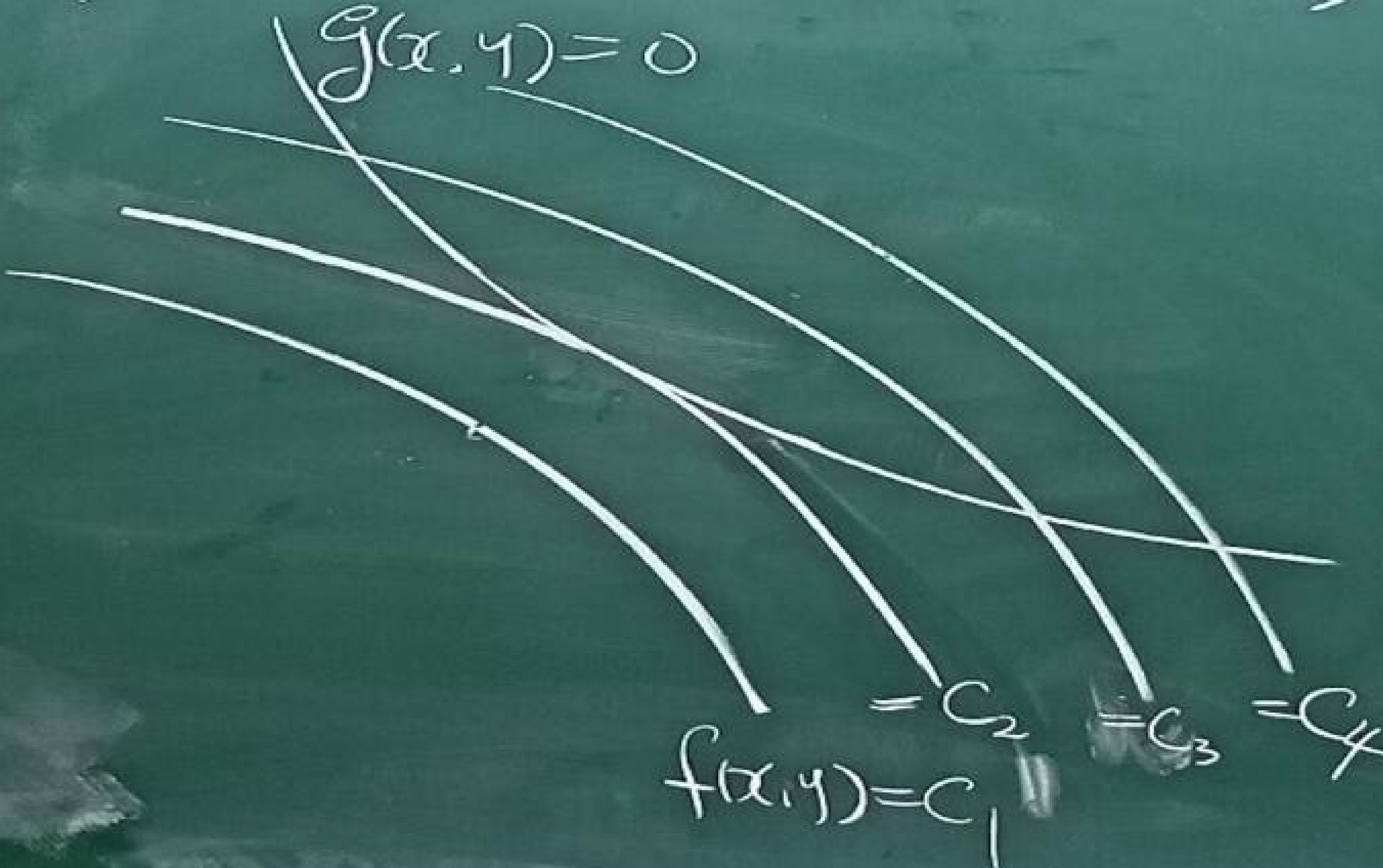
Constrained optimization
(Lagrangian Multiplier)

<u>Goal</u> Find local min/max of $f(x,y)$ on $\{g(x,y)=0\}$



Local extreme at (x_0, y_0)

$$\Rightarrow \begin{cases} g(x,y)=0 \\ f(x,y)=c \end{cases} \text{ are tangent at } (x_0,y_0)$$

$$\Rightarrow \nabla f(x_0,y_0) \parallel \nabla g(x_0,y_0)$$

$$\Rightarrow \nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$$

for some constant λ

$\Rightarrow$ Solve (x_0,y_0), λ

from $\begin{cases} g(x_0,y_0)=0 \\ \nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0) \end{cases}$

3 equations, 3 unknowns

Eg 1: Find the nearest point to origin on $(x-\frac{1}{2})^2+\frac{y^2}{4}=1$

<u>Sol</u>: Minimize $f(x,y)=(x-0)^2+(y-0)^2$

subject to $g(x,y)=(x-\frac{1}{2})^2+\frac{y^2}{4}-1=0$

$$\begin{cases} (x_0-\frac{1}{2})^2+\frac{y_0^2}{4}=1 & (g=0) & (1) \\ 2x_0=2\lambda(x_0-\frac{1}{2}) & (f_x=\lambda g_x) & (2) \\ 2y_0=\lambda\cdot\frac{y_0}{2} & (f_y=\lambda g_y) & (3) \end{cases}$$

$(3)\Longrightarrow y_0=0$ or $\lambda=4$

<u>Case A</u>: $y_0=0$

$$\begin{cases} (x_0-\frac{1}{2})^2=1 & (A_1) \\ 2x_0=2\lambda(x_0-\frac{1}{2}) & (A_2) \end{cases}$$

$(A_1) \Rightarrow x_0 = \boxed{\frac{-1}{2}}$ or $\boxed{\frac{3}{2}}$

"+"$(A_2) \Rightarrow \lambda = \boxed{\frac{1}{2}}$ $\boxed{3}$

<u>Case B</u> $\lambda = 4$

"+"(2) $\Rightarrow x_0 = \frac{2}{3}$

"+"(1) $\Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$

Overall, we have

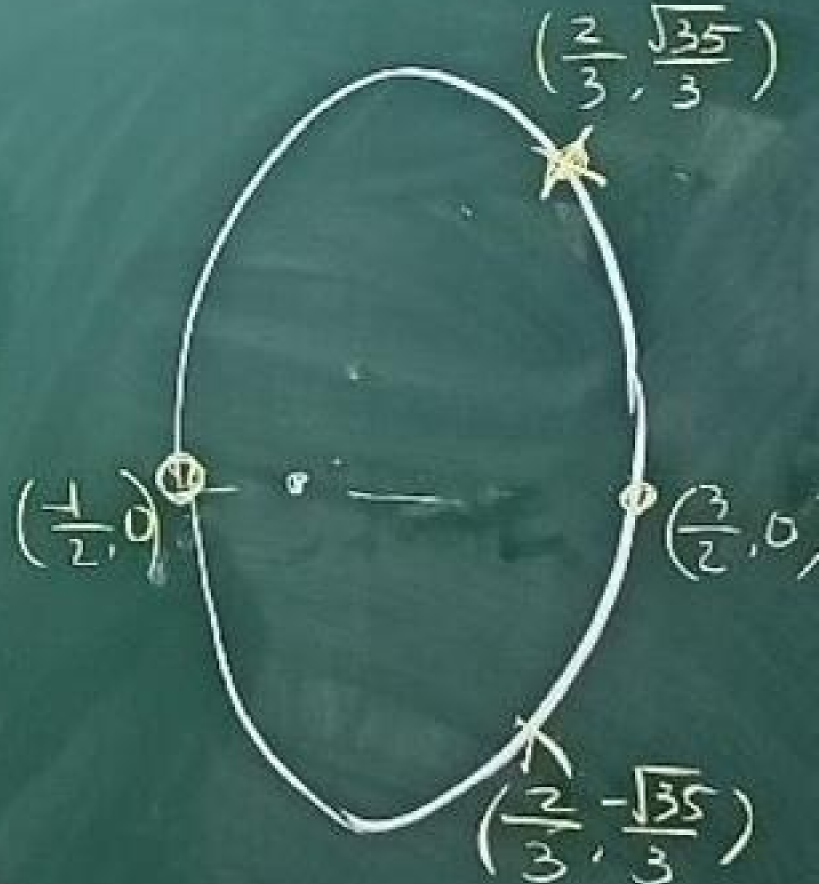
nearest

$(x_0, y_0) = \underline{\left(\frac{-1}{2}, 0\right)}, \left(\frac{3}{2}, 0\right)$

$\left(\frac{2}{3}, \frac{\sqrt{35}}{3}\right), \left(\frac{2}{3}, \frac{-\sqrt{35}}{3}\right)$

$f(x_0, y_0)$ (= distance^2)

$= \underbrace{\frac{1}{4}}_{\text{abs min}}, \frac{9}{4}, \underbrace{\frac{13}{3}, \frac{13}{3}}_{\text{abs max}}$



Recall: Find local extremes of $f(x,y)$ subject to $g(x,y)=0$

$\Rightarrow$ Solve for (x_0,y_0), λ from

$$\begin{cases} g(x_0,y_0)=0 \\ \nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0) \end{cases}$$

$\underline{\underline{3}}$ equations, $\underline{\underline{3}}$ unknowns

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§ Find local extremes of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$

$\Rightarrow$ Solve for (x_0,y_0,z_0), λ from

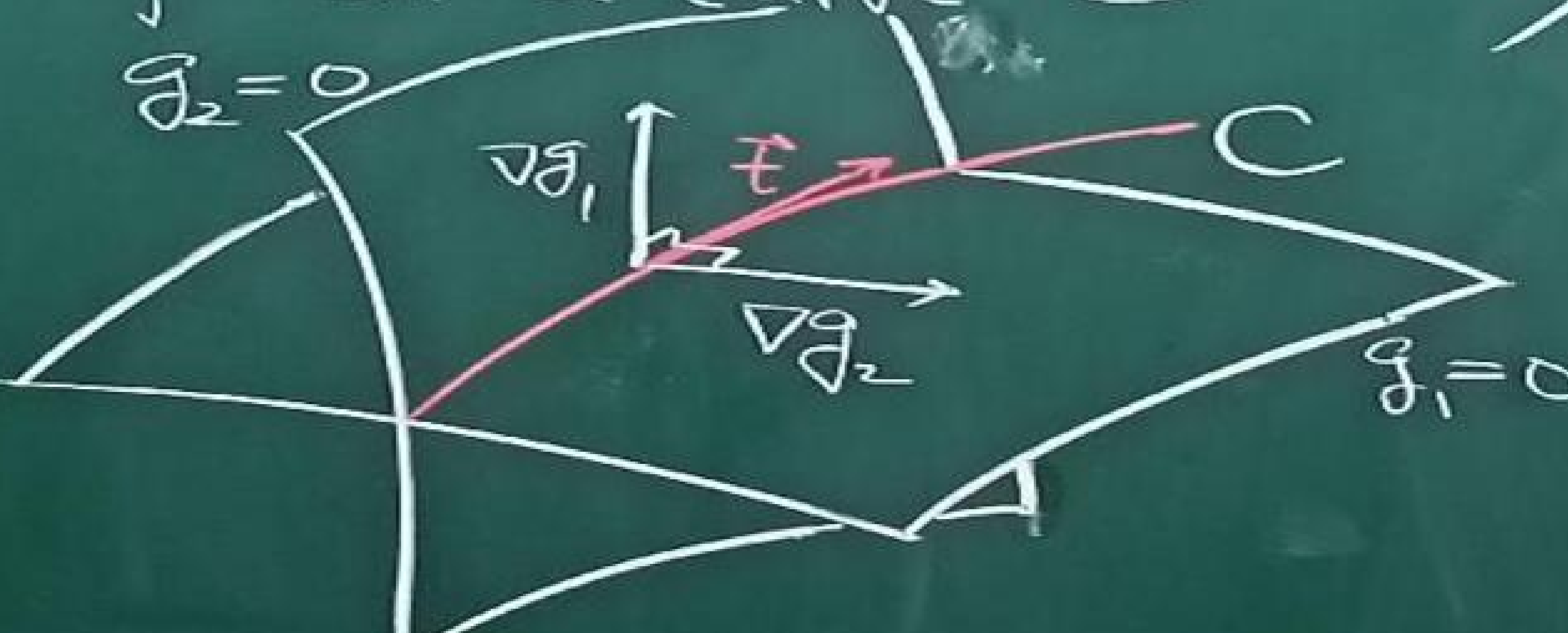
$$\begin{cases} g(x_0,y_0,z_0)=0 \\ \nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0) \end{cases}$$

$\underline{\underline{4}}$ equations, $\underline{\underline{4}}$ unknowns

Find local extremes of $f(x,y,z)$ subject to

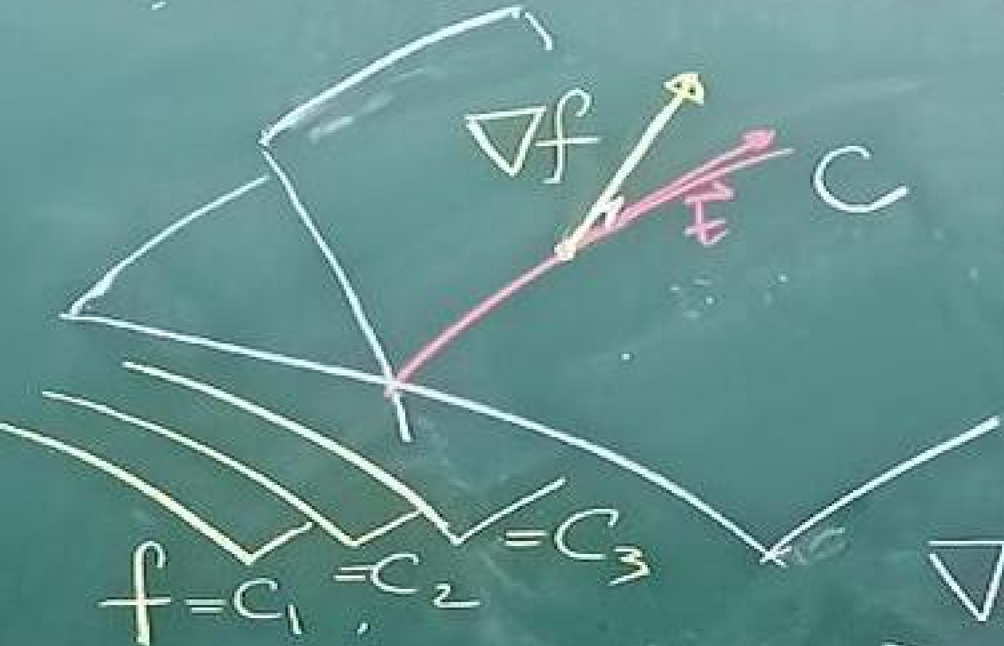
$$\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$

(Find local extremes of f on a curve C)



(Figure 14.38)

At a local extreme point



$\vec{t}$: tangent vector of C at (x_0, y_0, z_0)

$C \subseteq \{g_1 = 0\}$

$\nabla g_1(x_0, y_0, z_0) \perp \{g_1 = 0\}$

$\Rightarrow \nabla g_1(x_0, y_0, z_0) \perp \vec{t}$

Similarly $\nabla g_2(x_0, y_0, z_0) \perp \vec{t}$

Moreover, at a local extreme point C is tangent to $\{f(x, y, z) = c\}$

$\Rightarrow \vec{t} \perp \nabla f, \nabla g_1, \nabla g_2$ at (x_0, y_0, z_0)

$\Rightarrow \nabla f, \nabla g_1, \nabla g_2$ are co-plane at (x_0, y_0, z_0)

$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ at (x_0, y_0, z_0)

Summary:

Solve (x_0, y_0, z_0), λ_1, λ_2 from

$$g_1(x_0, y_0, z_0) = 0$$

$$g_2(x_0, y_0, z_0) = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\left(\begin{aligned} \partial_x f &= \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2 \\ \partial_y f &= \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2 \\ \partial_z f &= \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2 \end{aligned}\right)$$

$\underline{\underline{5}}$ equations, $\underline{\underline{5}}$ unknowns

Eg 1 Find nearest point to origin on the curve $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

Solve: Minimize $f(x,y,z)=x^2+y^2+z^2$

subject to $\begin{cases} g_1 = x+y+z=1 \\ g_2 = x^2+y^2=1 \end{cases}$

$$x+y+z=1 \quad \text{—— ①}$$

$$x^2+y^2=1 \quad \text{—— ②}$$

$$2x=\lambda_1 1+\lambda_2 \cdot 2x \quad \text{— ③}$$

$$2y=\lambda_1 1+\lambda_2 \cdot 2y \quad \text{— ④}$$

$$2z=\lambda_1 \cdot 1 \quad \text{— ⑤}$$

$$(5) \longrightarrow (3), (4)$$

$$\Rightarrow \begin{array}{l} x(1-\lambda_2) = z \\ y(1-\lambda_2) = z \end{array}$$

$$\Rightarrow (a)\ \lambda_2 = 1,\quad z = 0$$

or $(b)\ \lambda_2 \neq 1,\quad x = y$

Case (a)

$$(1)(2) \Rightarrow \begin{cases} x+y = 1 \\ x^2+y^2 = 1 \end{cases}$$

$$\Rightarrow (x,y,z) = \boxed{\begin{array}{c}(1,0,0)\\(0,1,0)\end{array}}$$

Case (b) $(x=y)$

$$(2) \Longrightarrow (x,y) = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$z = 1-x-y = 1 \mp \sqrt{2}$$

$$(1,0,0),\ (0,1,0),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$$

$$f = \underbrace{1,\ 1}_{\text{abs min}} \qquad \underbrace{4-2\sqrt{2}}_{\text{local max}}, \qquad \underbrace{4+2\sqrt{2}}_{\text{abs max}}$$

Taylor polynomial of

$f(x,y)$ centered at (a,b)

$x = a + h$

$y = b + k$

(x, y)

(a, b)

(1

Let $F(t) = f(a+th, b+tk)$

$h = x - a, \; k = y - b, \; t \in \mathbb{R}$

$F(0) = f(a,b), \; F(1) = f(x,y)$

Assume f and all partial derivatives of f are continuous everywhere.

Taylor's Thm for $F(t)$

$$F(t) = P_n(t) + R_n(t)$$

(Want $F(1) = P_n(1) + R_n(1)$) (*)

where $P_n(t) = \sum_{\ell=0}^{n} \frac{F^{(\ell)}(0)}{\ell!} t^\ell$

$$R_n(t) = \frac{F^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

for some c between 0 and t

Take $t = 1$

$$P_n(1) = \sum_{\ell=0}^{n} \frac{F^{(\ell)}(0)}{\ell!}$$

$$R_n(1) = \frac{F^{(n+1)}(c)}{(n+1)!}, \quad 0 < c < 1$$

$$F^{(k)}(0) = ?$$

$$F'(t) = \frac{d}{dt} f(x(t), y(t))$$

where $x(t) = a + th$, $y(t) = b + tk$

$$= f_x(x(t), y(t)) \cdot \underbrace{x'(t)}_{=h} + f_y(x(t), y(t)) \cdot \underbrace{y'(t)}_{=k}$$

$$= (h\partial_x + k\partial_y) f(x(t), y(t))$$

$$\therefore \text{"} \frac{d}{dt} F = (h\partial_x + k\partial_y) f \text{"}$$

$$F''(t) = \frac{d}{dt} F'(t) = \frac{d}{dt} \overbrace{(h\partial_x + k\partial_y) f}^{f_1(x(t), y(t))}$$

$$= \partial_x(\underbrace{(h\partial_x + k\partial_y) f}_{f_1}) \cdot \frac{dx}{dt} + \partial_y((h\partial_x + k\partial_y) f) \cdot \frac{dy}{dt}$$

$$= h\partial_x((h\partial_x + k\partial_y) f) + k\partial_y((h\partial_x + k\partial_y) f)$$

$$= (h\partial_x + k\partial_y)^2 f(x(t), y(t))$$

In Summary

$$F^{(\ell)}(t) = (h\partial_x + k\partial_y)^{\ell} f(a+th, b+tk)$$

for example:

$$F^{(3)} = (h^3\partial_x^3 + 3h^2k\partial_x^2\partial_y + 3hk^2\partial_x\partial_y^2 + k^3\partial_y^3) f$$

$$\overset{(*)}{\Longrightarrow} f(x,y) = \tilde{P}_n(x,y) + \tilde{R}_n(x,y)$$

$$\tilde{P}_n(x,y) = P_n(1) = \sum_{\ell=0}^{n} \frac{(h\partial_x + k\partial_y)^{\ell}}{\ell!} f\Big|_{(a,b)}$$

$$\tilde{R}_n(x,y) = R_n(1)$$

$$= \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch, b+ck)$$

$$0 < c < 1$$

Recall:

Let $F(t) = f(a+th, b+tk)$

$\Rightarrow F^{(\ell)}(t) = (h\partial_x + k\partial_y)^\ell f(a+th, b+tk)$

or symbolically $\left(\frac{d}{dt}\right)^\ell = (h\partial_x + k\partial_y)^\ell$

Similarly for $g(x,y,z)$

Let $G(t) = g(x_0+t\Delta x, y_0+t\Delta y, z_0+t\Delta z)$

$\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta z = z - z_0$

$$\left(\frac{d}{dt}\right)^\ell G = (\Delta x\,\partial_x + \Delta y\,\partial_y + \Delta z\,\partial_z)^\ell g$$

$$\tilde{P}_n = \sum_{\ell=0}^{n} \frac{(\Delta x\partial_x + \Delta y\partial_y + \Delta z\partial_z)^\ell}{\ell!} g(x_0, y_0, z_0)$$

$$\tilde{R}_n = \frac{(\cdots)^{n+1}}{(n+1)!} g(x_0+c\Delta x, y_0+c\Delta y, z_0+c\Delta z) \quad 0<c<1$$

Eg1 Error of linearization for $f(x,y)$

$$L(x,y) = f(x_0,y_0) + \underbrace{\partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)}_{(\Delta x \partial_x + \Delta y \partial_y)^1 f(x_0,y_0)}$$

$$\left(= \tilde{P}_1(x,y)\right)$$

$$f(x,y) = \tilde{P}_1(x,y) + \tilde{R}_1(x,y)$$

$$\therefore \text{error} = f(x,y) - L(x,y) = \tilde{R}_1(x,y)$$

$$= \frac{(\Delta x \partial_x + \Delta y \partial_y)^2}{2!} f(x_0 + c\Delta x,\ y_0 + c\Delta y)$$

$$\therefore |\text{error}| \leq \frac{1}{2}\Big(\underbrace{|\Delta x^2 \partial_x^2 f| + |2\Delta x \Delta y \partial_x \partial_y f| + |\Delta y^2 \partial_y^2 f|}_{\text{at } (x_0 + c\Delta x,\ y_0 + c\Delta y)}\Big)$$

$$\leq \frac{M}{2}(|\Delta x| + |\Delta y|)^2, \text{ if } |f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$$

Similarly

$$|f(x,y,z)-L(x,y,z)| \leq \frac{M}{2}(|\Delta x|+|\Delta y|+|\Delta z|)^2$$

if all $|$2nd derivatives$| \leq M$ in the region

Eg2. Local min/max of f

$$f(x,y) = \underbrace{f(x_0,y_0)}_{l=0} + \underbrace{\Delta_1}_{l=1} + \underbrace{\Delta_2}_{l=2} + \underbrace{\Delta_3}_{\tilde{R}_2}$$

At a critical point, $\Delta_1 = 0$

$\Delta_2 =$ leading term: $O(\Delta x^2, \Delta x\Delta y, \Delta y^2)$

$$\Delta_3 = \tilde{R}_2(x,y) = \frac{(\Delta x \partial_x + \Delta y \partial_y)^3}{3!} f(x_0+c\Delta x, y_0+c\Delta y)$$

$$\therefore |\Delta_3| \leq \frac{1}{3!}\left(M|\Delta x|^3 + 3M|\Delta x|^2|\Delta y| + 3M|\Delta x||\Delta y|^2 + M|\Delta y|^3\right)$$

$$= \frac{M}{3!}(|\Delta x|+|\Delta y|)^3$$

if $|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}| \leq M$