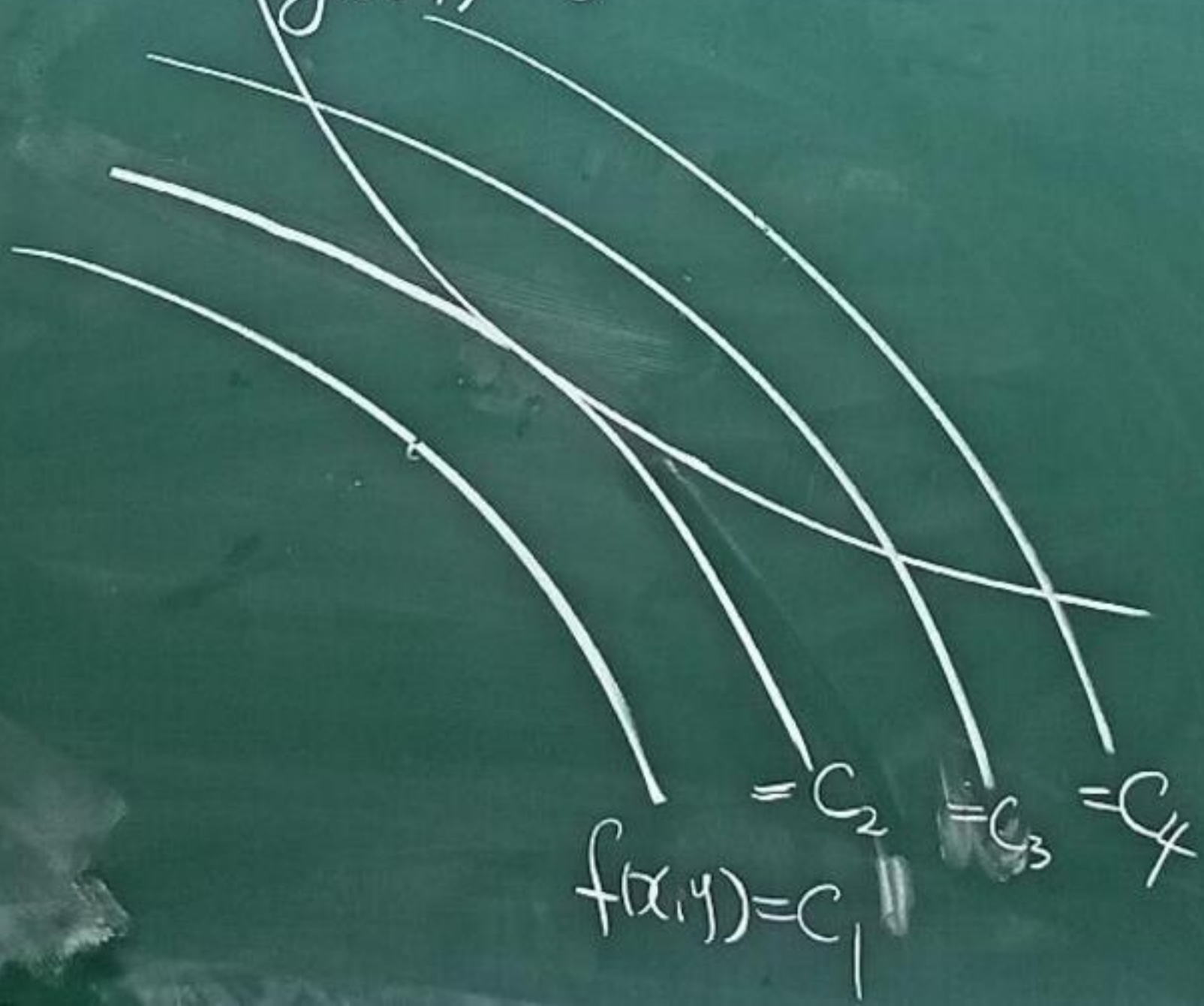


Constrained optimization (Lagrangian Multiplier)

Goal Find local min/max
of $f(x, y)$ on $\{g(x, y) = 0\}$

$$g(x, y) = 0$$



Local extreme at (x_0, y_0)

$\Rightarrow \begin{cases} g(x, y) = 0 \\ f(x, y) = c \end{cases}$ are tangent at (x_0, y_0)

$\Rightarrow \nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$

$\Rightarrow \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$
for some constant λ

\Rightarrow Solve $(x_0, y_0), \lambda$

from $\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$

3 equations, 3 unknowns

Ex 1: Find the nearest

point to origin on $(x - \frac{1}{2})^2 + \frac{y^2}{4} = 1$

Sol: Minimize $f(x, y) = (x - 0)^2 + (y - 0)^2$

Subject to $g(x, y) = (x - \frac{1}{2})^2 + \frac{y^2}{4} - 1 = 0$

$$\begin{cases} (1) & \left\{ \begin{array}{l} (x_0 - \frac{1}{2})^2 + \frac{y_0^2}{4} = 1 \\ 2x_0 = 2\lambda(x_0 - \frac{1}{2}) \\ 2y_0 = \lambda \cdot \frac{y_0}{2} \end{array} \right. \quad \begin{array}{l} (g=0) \\ (f_x = \lambda g_x) \\ (f_y = \lambda g_y) \end{array} \end{cases}$$

$$(3) \Rightarrow y_0 = 0 \text{ or } \lambda = 4$$

Case A: $y_0 = 0$

$$\begin{cases} (x_0 - \frac{1}{2})^2 = 1 & (A_1) \\ 2x_0 = 2\lambda(x_0 - \frac{1}{2}) & (A_2) \end{cases}$$

$$(A_1) \Rightarrow x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \propto \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

$$"(A_2) \Rightarrow \lambda = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Case B $\lambda = 4$

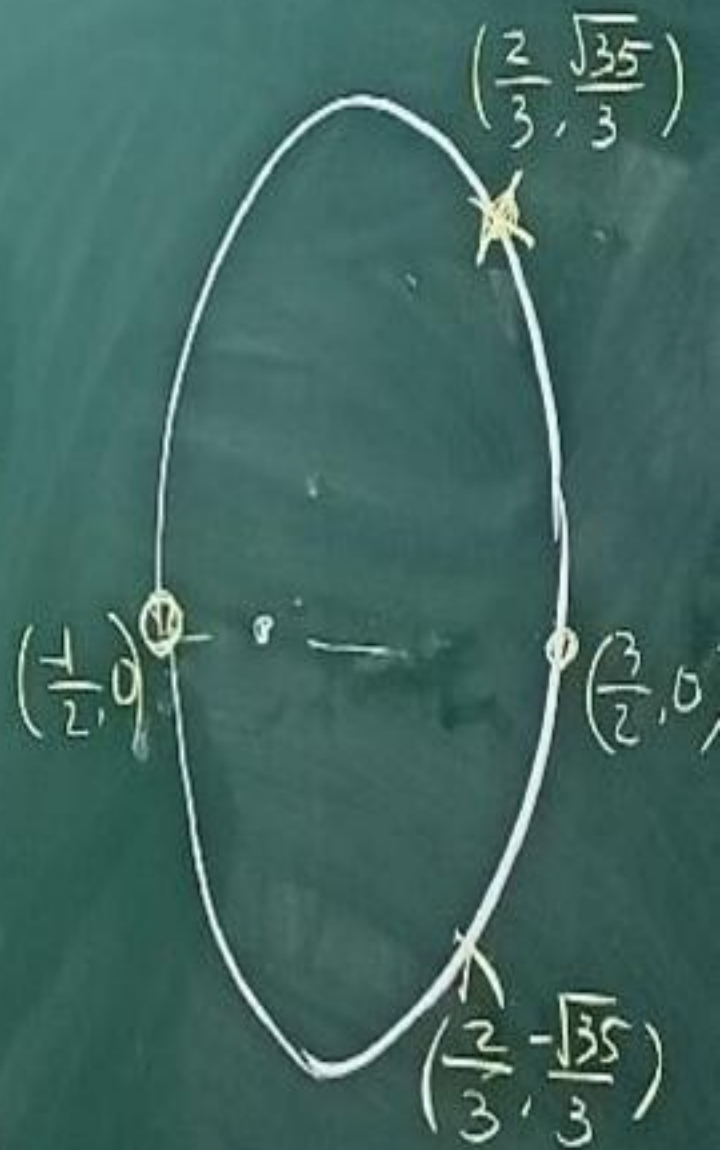
$$"(2) \Rightarrow x_0 = \frac{2}{3}$$

$$"(1) \Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$$

Overall, we have

$$(x_0, y_0) = \left(\frac{-1}{2}, 0 \right), \left(\frac{3}{2}, 0 \right)$$

$$\left(\frac{2}{3}, \frac{\sqrt{35}}{3} \right), \left(\frac{2}{3}, -\frac{\sqrt{35}}{3} \right)$$



$f(x_0, y_0)$ (= distance²)

$$= \frac{1}{4}, \frac{9}{4}, \frac{13}{3}, \frac{13}{3}$$

abs min

abs max

Recall: Find local extremes of

$f(x, y)$, subject to $g(x, y) = 0$

\Rightarrow Solve for $(x_0, y_0), \lambda$ from

$$\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$$

3 equations, 3 unknowns

\hookrightarrow Find local extremes of $f(x, y, z)$

subject to the constraint $g(x, y, z) = 0$

\Rightarrow Solve for $(x_0, y_0, z_0), \lambda$ from

$$\begin{cases} g(x_0, y_0, z_0) = 0 \\ \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \end{cases}$$

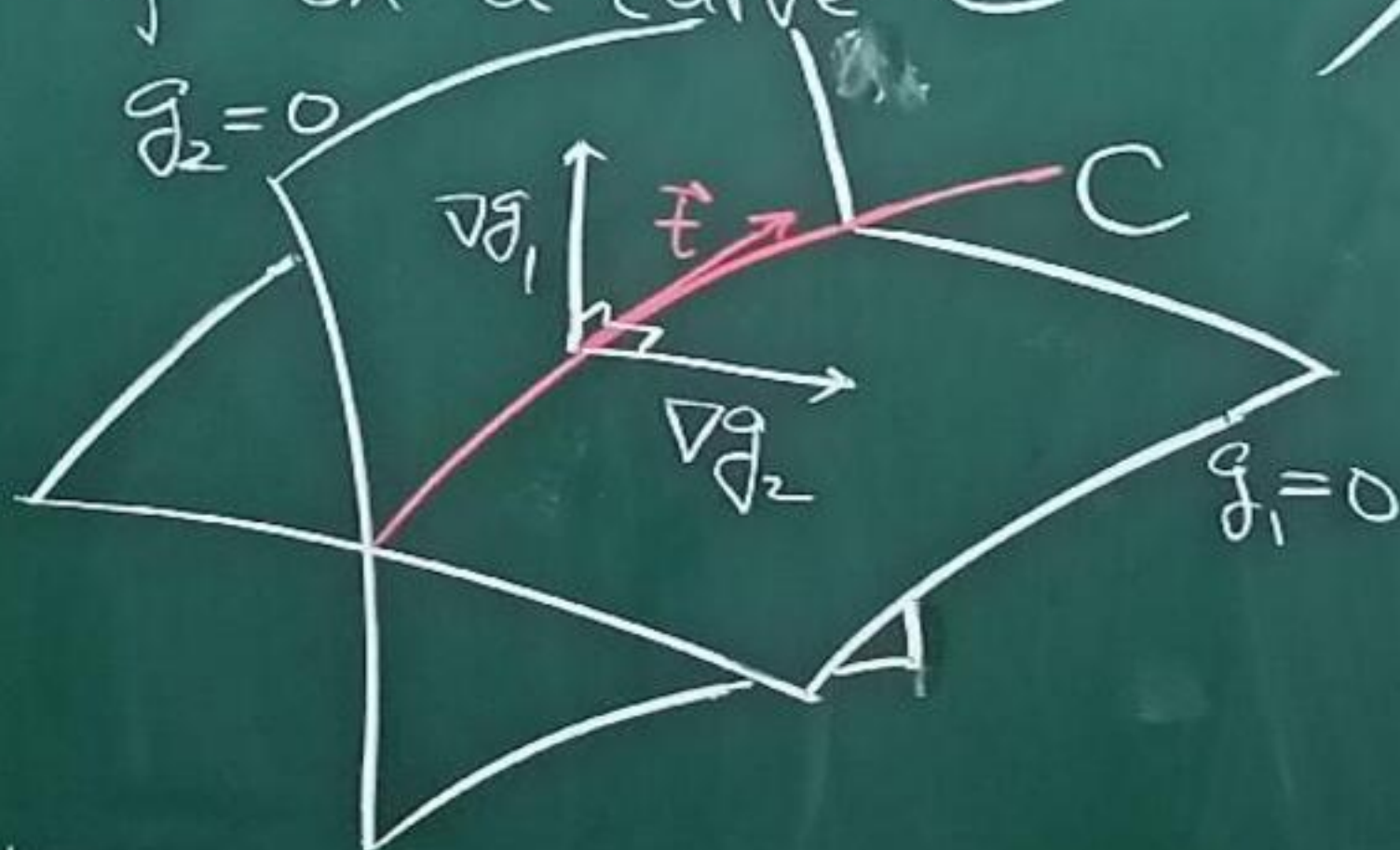
4 equations, 4 unknowns

Find local extremes of

$f(x, y, z)$ subject to

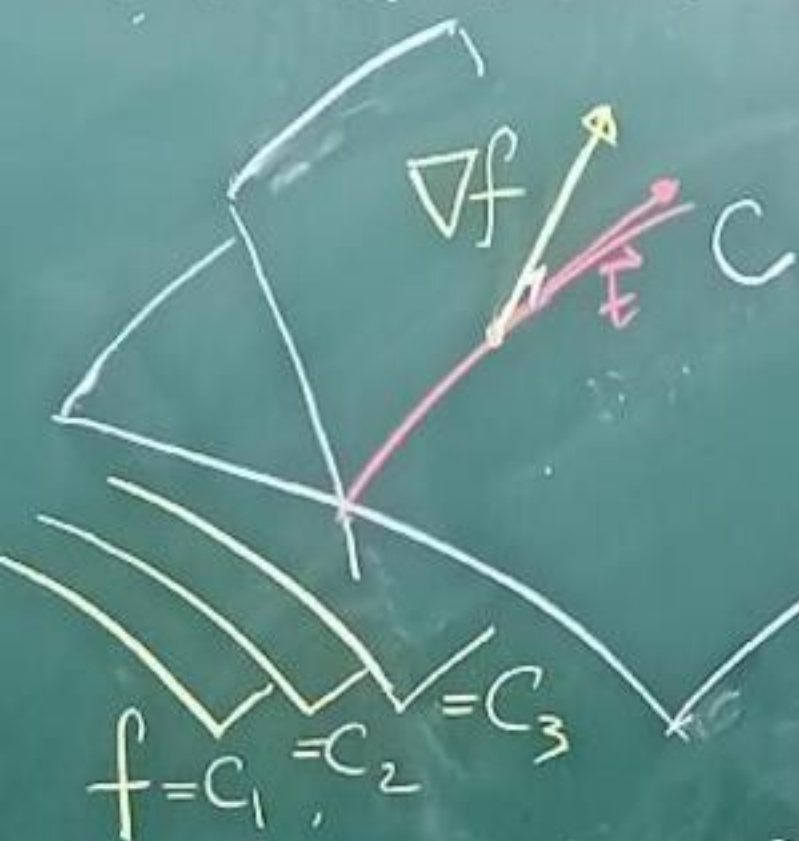
$$\begin{cases} g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases}$$

(Find local extremes of f on a curve C)



(Figure 14.38)

At a local extreme point



\vec{t} : tangent vector
of C at (x_0, y_0, z_0)

$$C \subseteq \{g_1 = 0\}$$

$$\nabla g_1(x_0, y_0, z_0) \perp \{g_1 = 0\}$$

$$\Rightarrow \nabla g_1(x_0, y_0, z_0) \perp \vec{t}$$

Similarly $\nabla g_2(x_0, y_0, z_0) \perp \vec{t}$

Moreover, at a local extreme point

C is tangent to $\{f(x, y, z) = c\}$

$$\Rightarrow \vec{t} \perp \nabla f, \nabla g_1, \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\Rightarrow \nabla f, \nabla g_1, \nabla g_2 \text{ are coplanar at } (x_0, y_0, z_0)$$

$$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

Summary:

Solve (x_0, y_0, z_0) , λ_1, λ_2 from

$$g_1(x_0, y_0, z_0) = 0$$

$$g_2(x_0, y_0, z_0) = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\left(\begin{array}{l} \partial_x f = \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2 \\ \partial_y f = \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2 \\ \partial_z f = \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2 \end{array} \right)$$

5 equations, 5 unknowns

Eg 1 Find nearest point

to origin on the curve $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

Solve: Minimize $f(x, y, z) = x^2 + y^2 + z^2$

subject to $\begin{cases} g_1 = x+y+z=1 \\ g_2 = x^2+y^2=1 \end{cases}$

$$x+y+z=1 \quad \text{--- (1)}$$

$$x^2+y^2=1 \quad \text{--- (2)}$$

$$2x = \lambda_1 \cdot 1 + \lambda_2 \cdot 2x \quad \text{--- (3)}$$

$$2y = \lambda_1 \cdot 1 + \lambda_2 \cdot 2y \quad \text{--- (4)}$$

$$2z = \lambda_1 \cdot 1 \quad \text{--- (5)}$$

$$\textcircled{5} \rightarrow \textcircled{3}, \textcircled{4}$$

$$\Rightarrow x(1-\lambda_2) = z$$

$$y(1-\lambda_2) = z$$

$$\Rightarrow \textcircled{a} \lambda_2 = 1, z = 0$$

$$\text{or } \textcircled{b} \lambda_2 \neq 1, x = y$$

Case (a)

$$\textcircled{1} \textcircled{2} \Rightarrow \begin{cases} x+y=1 \\ x^2+y^2=1 \end{cases}$$

$$\Rightarrow (x, y, z) = \begin{matrix} (1, 0, 0) \\ (0, 1, 0) \end{matrix}$$

Case (b) ($x=y$)

$$\textcircled{2} \Rightarrow (x, y) = \pm \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$z = 1 - x - y = 1 \mp \sqrt{2}$$

$$\underline{(1, 0, 0), (0, 1, 0)}, \underline{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2} \right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1+\sqrt{2} \right)}$$

$$f = \underbrace{1, 1}$$

abs min


$$\underline{4-2\sqrt{2}}$$

local max

$$\underline{4+2\sqrt{2}}$$

abs max

Taylor polynomial of
 $f(x, y)$ centered at (a, b)

$$\begin{aligned}x &= a + h \\y &= b + k\end{aligned}$$


(a, b) (x, y)

Let $F(t) = f(a + th, b + tk)$
 $h = x - a, k = y - b, t \in \mathbb{R}$

$$F(0) = f(a, b), F(1) = f(x, y)$$

Assume f and all partial derivatives of f are continuous everywhere.

Taylor's Thm for $F(t)$

$$F(t) = P_n(t) + R_n(t)$$

$$\text{(Want } F(1) = P_n(1) + R_n(1) \text{)} \quad (*)$$

$$\text{where } P_n(t) = \sum_{l=0}^n \frac{F^{(l)}(0)}{l!} t^l$$

$$R_n(t) = \frac{F^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

for some c between 0 and t

Take $t=1$

$$P_n(1) = \sum_{l=0}^n \frac{F^{(l)}(0)}{l!}$$

$$R_n(1) = \frac{F^{(n+1)}(c)}{(n+1)!}, \quad 0 < c < 1$$

$$F^{(k)}(0) = ?$$

$$F'(t) = \frac{d}{dt} f(x(t), y(t))$$

$$\text{where } x(t) = a + th, \quad y(t) = b + tk$$

$$= f_x(x(t), y(t)) \cdot \overset{h}{x'(t)} + f_y(x(t), y(t)) \cdot \overset{k}{y'(t)}$$

$$= (h\partial_x + k\partial_y) f(x(t), y(t))$$

$$\therefore \frac{d}{dt} F = (h\partial_x + k\partial_y) f \quad \overset{f_1(x(t), y(t))}{f}$$

$$F''(t) = \frac{d}{dt} F'(t) = \frac{d}{dt} (h\partial_x + k\partial_y) f$$

$$= \partial_x((h\partial_x + k\partial_y) f) \cdot \frac{dx}{dt} + \partial_y((h\partial_x + k\partial_y) f) \cdot \frac{dy}{dt}$$

$$= h\partial_x((h\partial_x + k\partial_y) f) + k\partial_y((h\partial_x + k\partial_y) f)$$

$$= (h\partial_x + k\partial_y)^2 f(x(t), y(t))$$

In Summary

$$F^{(l)}(t) = (h\alpha_x + k\alpha_y)^l \cdot f(a+th, b+tk)$$

for example:

$$F^{(3)} = (h^3\alpha_x^3 + 3hk^2\alpha_x^2\alpha_y + 3h^2k\alpha_x\alpha_y^2 + k^3\alpha_y^3) f$$

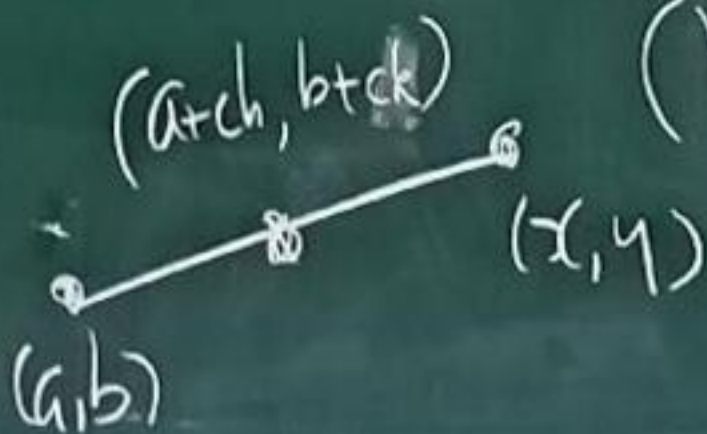
$$\stackrel{(*)}{\Rightarrow} f(x, y) = \tilde{P}_n(x, y) + \tilde{R}_n(x, y)$$

$$\tilde{P}_n(x, y) = P_n(1) = \sum_{l=0}^n \frac{(h\alpha_x + k\alpha_y)^l}{l!} f \Big|_{(a, b)}$$

$$\tilde{R}_n(x, y) = R_n(1)$$

$$= \frac{(h\alpha_x + k\alpha_y)^{n+1}}{(n+1)!} f(a+ch, b+ck)$$

$$0 < c < 1$$



Recall:

$$\text{Let } F(t) = f(a+th, b+tk)$$

$$\Rightarrow F^{(l)}(t) = (h\partial_x + k\partial_y)^l f(a+th, b+tk)$$

$$\text{or symbolically } \left(\frac{d}{dt}\right)^l = (h\partial_x + k\partial_y)^l$$

Similarly for $g(x, y, z)$

$$\text{Let } G(t) = g(x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z)$$

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta z = z - z_0$$

$$\left(\frac{d}{dt}\right)^l G = (\Delta x \partial_x + \Delta y \partial_y + \Delta z \partial_z)^l g$$

$$\tilde{P}_n = \sum_{l=0}^n \frac{t^l}{l!} (\Delta x \partial_x + \Delta y \partial_y + \Delta z \partial_z)^l g(x_0, y_0, z_0)$$

$$\tilde{R}_n = \frac{t^{n+1}}{(n+1)!} g^{(n+1)}(x_0 + c\Delta x, y_0 + c\Delta y, z_0 + c\Delta z) \quad 0 < c < 1$$

Ex 1 Error of linearization
for $f(x, y)$

$$L(x, y) = f(x_0, y_0) + \underbrace{\partial_x f(x_0, y_0)(x - x_0) + \partial_y f(x_0, y_0)(y - y_0)}_{(\Delta x \partial_x + \Delta y \partial_y) f(x_0, y_0)}$$

($= \tilde{P}_1(x, y)$)

$$f(x, y) = \tilde{P}_1(x, y) + \tilde{R}_1(x, y)$$

$$\therefore \text{error} = f(x, y) - L(x, y) = \tilde{R}_1(x, y)$$

$$= \frac{(\Delta x \partial_x + \Delta y \partial_y)^2}{2!} f(x_0 + c\Delta x, y_0 + c\Delta y)$$

$$\therefore |\text{error}| \leq \frac{1}{2} \left(|\Delta x^2 \partial_x^2 f| + |2\Delta x \Delta y \partial_x \partial_y f| + |\Delta y^2 \partial_y^2 f| \right)$$

$$\leq \frac{M}{2} (|\Delta x| + |\Delta y|)^2, \quad \text{at } (x_0 + c\Delta x, y_0 + c\Delta y) \text{ if } |f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$$

Similarly

$$|f(x, y, z) - L(x, y, z)| \leq \frac{M}{2} (|\Delta x| + |\Delta y| + |\Delta z|)^2$$

if all $|2^{\text{nd}} \text{ derivatives}| \leq M$ in the region

Ex 2. Local min/max of f

$$f(x, y) = \underbrace{f(x_0, y_0)}_{l=0} + \underbrace{\Delta_1}_{l=1} + \underbrace{\Delta_2}_{l=2} + \underbrace{\Delta_3}_{\tilde{R}_2}$$

At a critical point, $\Delta_1 = 0$

$\Delta_2 =$ leading term: $O(\Delta x^2, \Delta x \Delta y, \Delta y^2)$

$$\Delta_3 = \tilde{R}_2(x, y) = \frac{(\Delta x \partial_x + \Delta y \partial_y)^3}{3!} f(x_0 + c\Delta x, y_0 + c\Delta y)$$

$$\therefore |\Delta_3| \leq \frac{1}{3!} (M |\Delta x|^3 + 3M |\Delta x|^2 |\Delta y| + 3M |\Delta x| |\Delta y|^2 + M |\Delta y|^3)$$
$$= \frac{M}{3!} (|\Delta x| + |\Delta y|)^3$$

if $|f_{xxx}|, |f_{xyx}|, |f_{xxy}|, |f_{yyy}| \leq M$