

Eg1. find local extremes
of $f(x, y) = x^2 + y^2 - 4y + 9$

Sol. Method 1

$$f_x = 2x, \quad f_y = 2(y - 2)$$

\Rightarrow critical point = $(0, 2)$ only

$$f_{xx}(0, 2) = 2, \quad f_{xy}(0, 2) = 0$$

$$f_{yy}(0, 2) = 2 \Rightarrow D = -4$$

$$A > 0, \quad D < 0$$

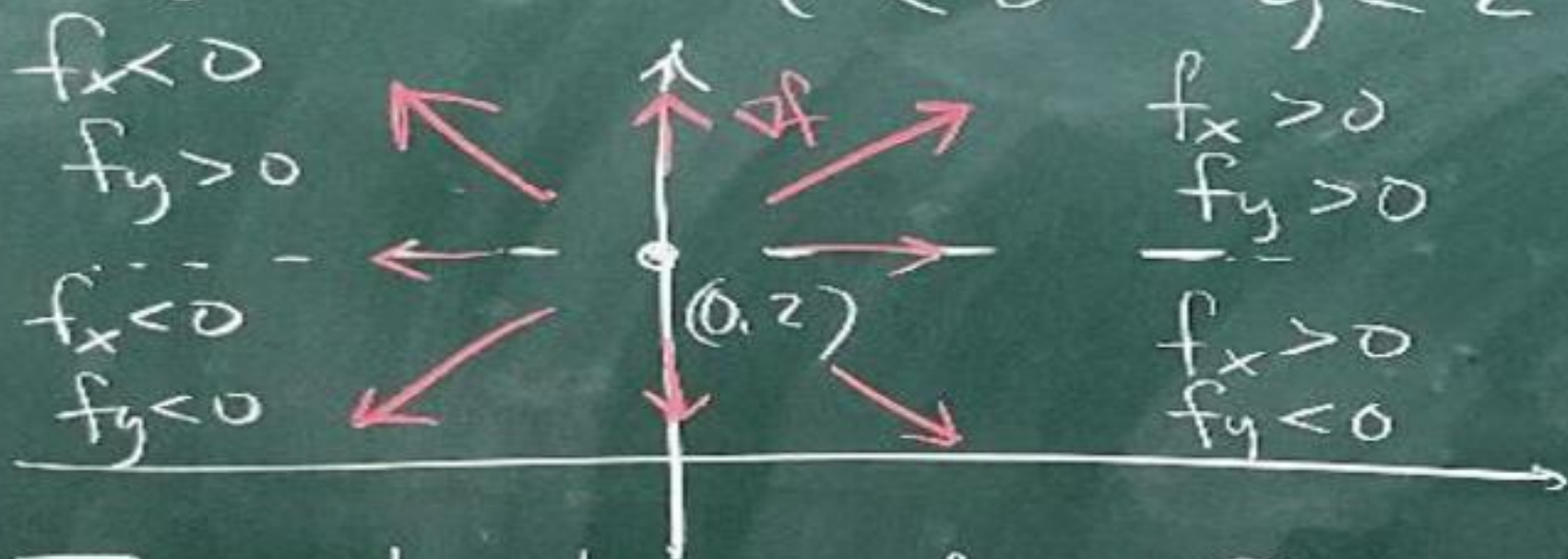
$$\Rightarrow \text{local min} = f(0, 2) = 5$$

local max: none

Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

$$f_y = 2(y-2) \begin{cases} > 0 & \text{if } y > 2 \\ < 0 & \text{if } y < 2 \end{cases}$$



From directions of ∇f near $(x_0, y_0) = (0, 2)$

$\Rightarrow f(0, 2)$ is a local min

Remark: Near a critical point
where $\nabla f(x_0, y_0) = (0, 0)$



∇f points outward
 \Rightarrow local min



∇f points inward
 \Rightarrow local max



∇f points inward
in some directions
and outward in some
directions \Rightarrow Not local min
Not local max
(eg. saddle points)

Ex 2: Find absolute extremes

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

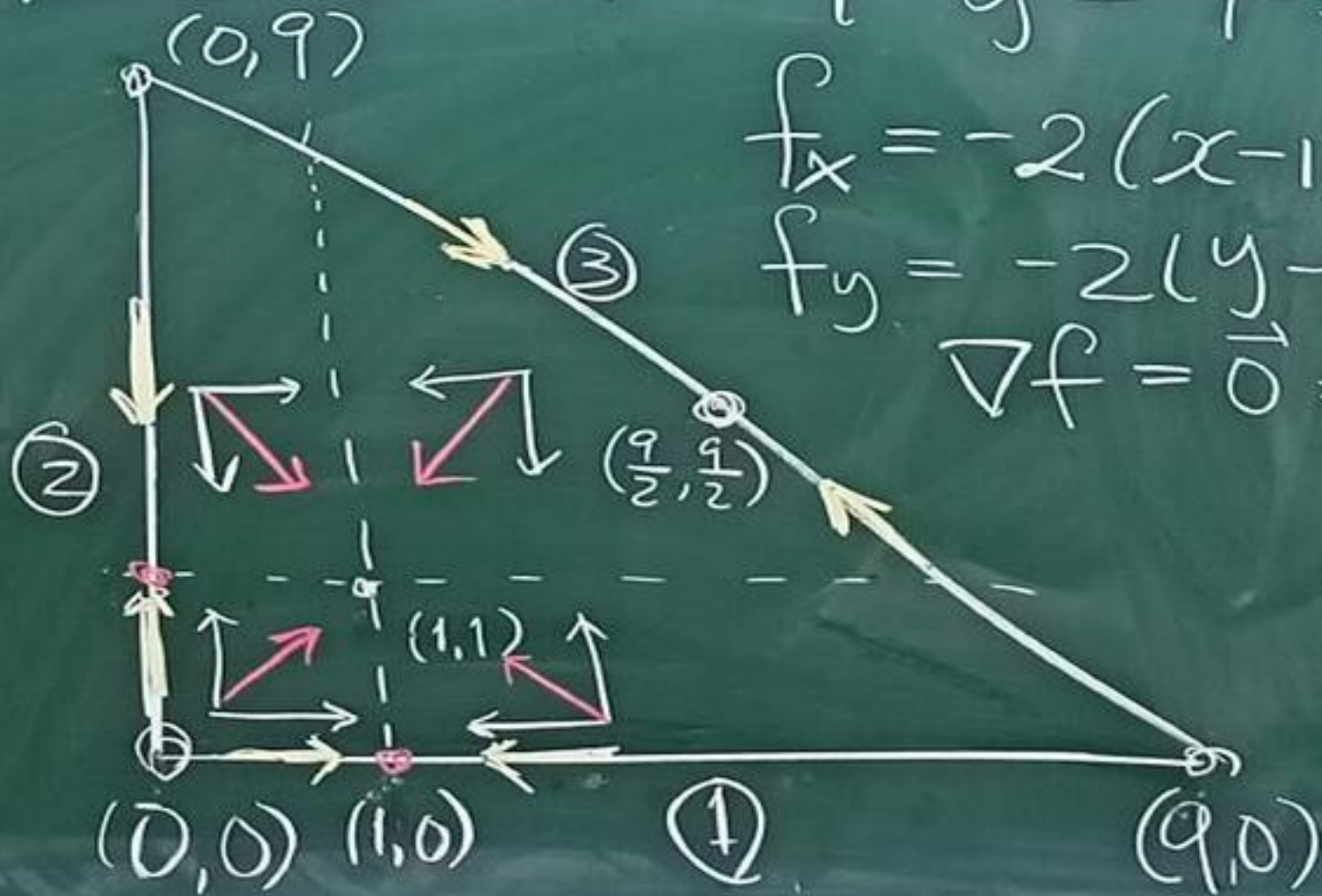
on the region
bounded by

$$\begin{cases} x=0 \\ y=0 \\ y=9-x \end{cases}$$

$$f_x = -2(x-1)$$

$$f_y = -2(y-1)$$

$$\nabla f = \vec{0} \Rightarrow (1, 1)$$



Method 1 (textbook)

Critical Point = (1, 1)

Compare $f(1, 1)$ with all values of f on boundary

Method 2 (Gradient Analysis)

Near (1, 1)



local max (1, 1)

on ① = $\begin{cases} y=0 \\ 0 \leq x \leq 9 \end{cases}$



local max = (1, 0)

local min = (0, 0), (9, 0)

on ②:

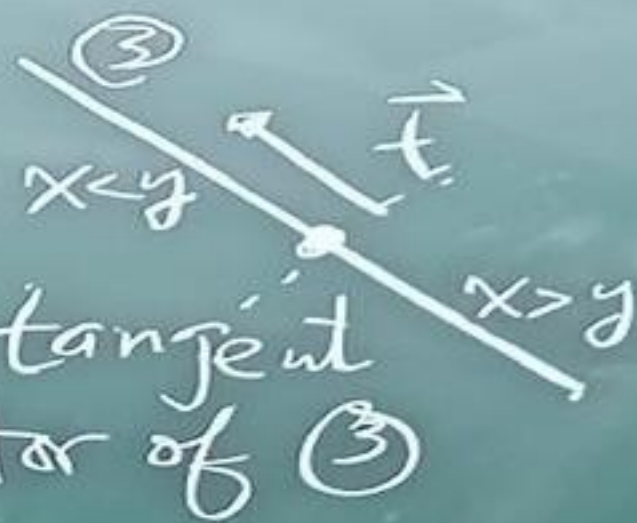


(0, 1)

local max = (0, 1)

local min = (0, 0), (0, 9)

On ③ = $\begin{cases} x+y=9 \\ 0 \leq x \leq 9 \end{cases}$



Let $\vec{t} = (-1, 1)$ = a tangent vector of ③

$$\nabla f \cdot \vec{t} \geq 0 \text{ ?}$$

$$\nabla f \cdot \vec{t} \leq 0 \text{ ?}$$

$$= (-2(x-1), -2(y-1)) \cdot (-1, 1)$$

$$= \sqrt{2}(x-y) \begin{cases} > 0 & \text{if } x > y \\ < 0 & \text{if } x < y \end{cases}$$



On ③, local max = $(\frac{9}{2}, \frac{9}{2})$

local min = $(9, 0), (0, 9)$

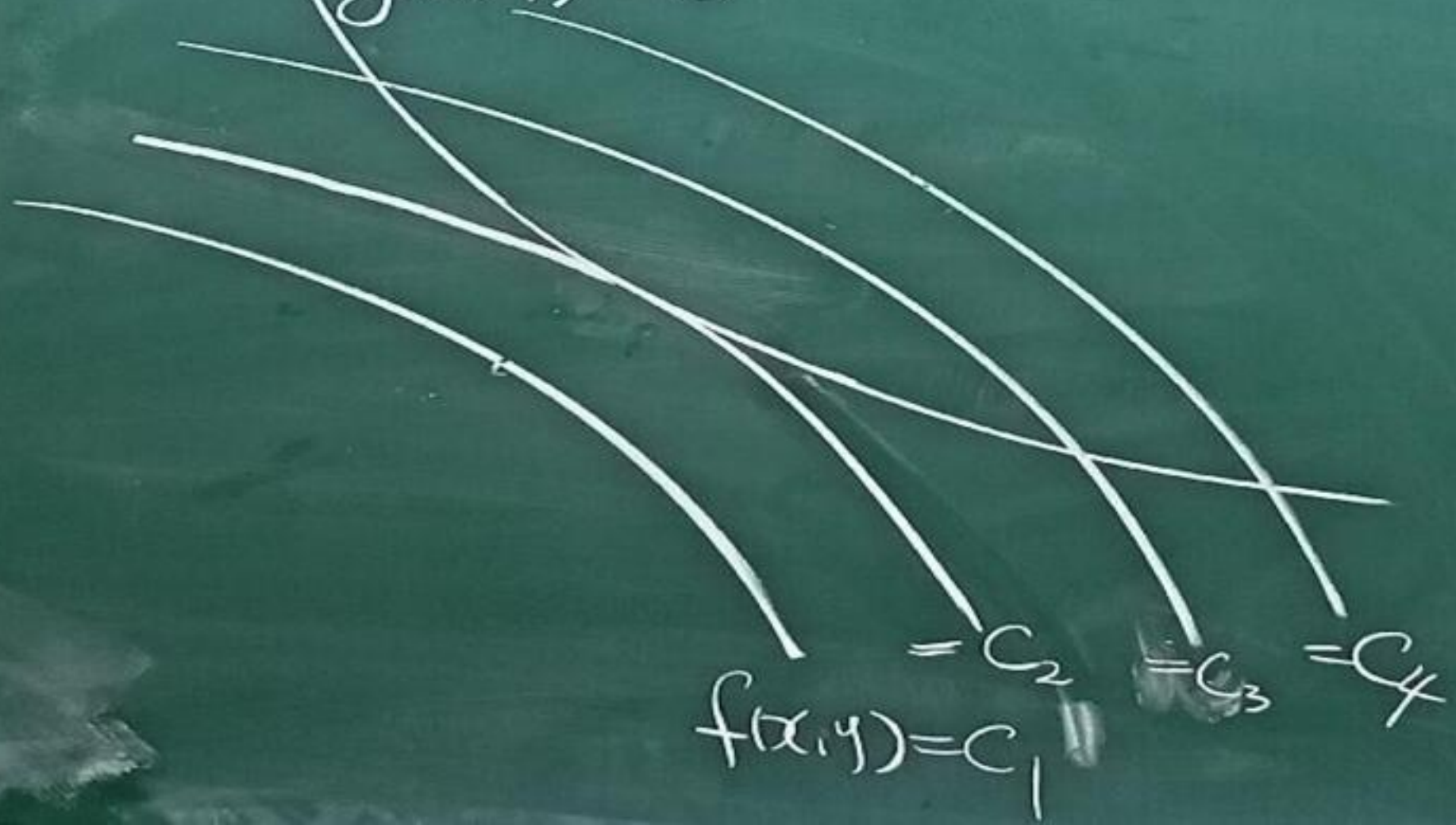
Absolute max = $f(1, 1)$

Abs. min: Compare $f(0, 0), f(0, 9), f(9, 0)$

Constrained optimization (Lagrangian Multiplier)

Goal Find local min/max
of $f(x, y)$ on $\{g(x, y) = 0\}$

$$g(x, y) = 0$$



Local extreme at (x_0, y_0)

$\Rightarrow \begin{cases} g(x, y) = 0 \\ f(x, y) = c \end{cases}$ are tangent at (x_0, y_0)

$\Rightarrow \nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$

$\Rightarrow \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$
for some constant λ

\Rightarrow Solve $(x_0, y_0), \lambda$

from $\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$

3 equations, 3 unknowns

Ex 1: Find the nearest

point to origin on $(x - \frac{1}{2})^2 + \frac{y^2}{4} = 1$

Sol: Minimize $f(x, y) = (x - 0)^2 + (y - 0)^2$

subject to $g(x, y) = (x - \frac{1}{2})^2 + \frac{y^2}{4} - 1 = 0$

$$\begin{aligned} (1) & \left\{ \begin{aligned} (x_0 - \frac{1}{2})^2 + \frac{y_0^2}{4} &= 1 & (g=0) \\ 2x_0 &= \lambda(x_0 - \frac{1}{2}) & (f_x = \lambda g_x) \\ 2y_0 &= \lambda \cdot \frac{y_0}{2} & (f_y = \lambda g_y) \end{aligned} \right. \end{aligned}$$

$$(3) \Rightarrow y_0 = 0 \text{ or } \lambda = 4$$

Case A: $y_0 = 0$

$$\begin{cases} (x_0 - \frac{1}{2})^2 = 1 & (A_1) \\ 2x_0 = 2\lambda(x_0 - \frac{1}{2}) & (A_2) \end{cases}$$

$$(A_1) \Rightarrow x_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \propto \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$"(A_2) \Rightarrow \lambda = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \propto \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Case B $\lambda = 4$

$$"(2) \Rightarrow x_0 = \frac{2}{3}$$

$$"(1) \Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$$

Overall, we have

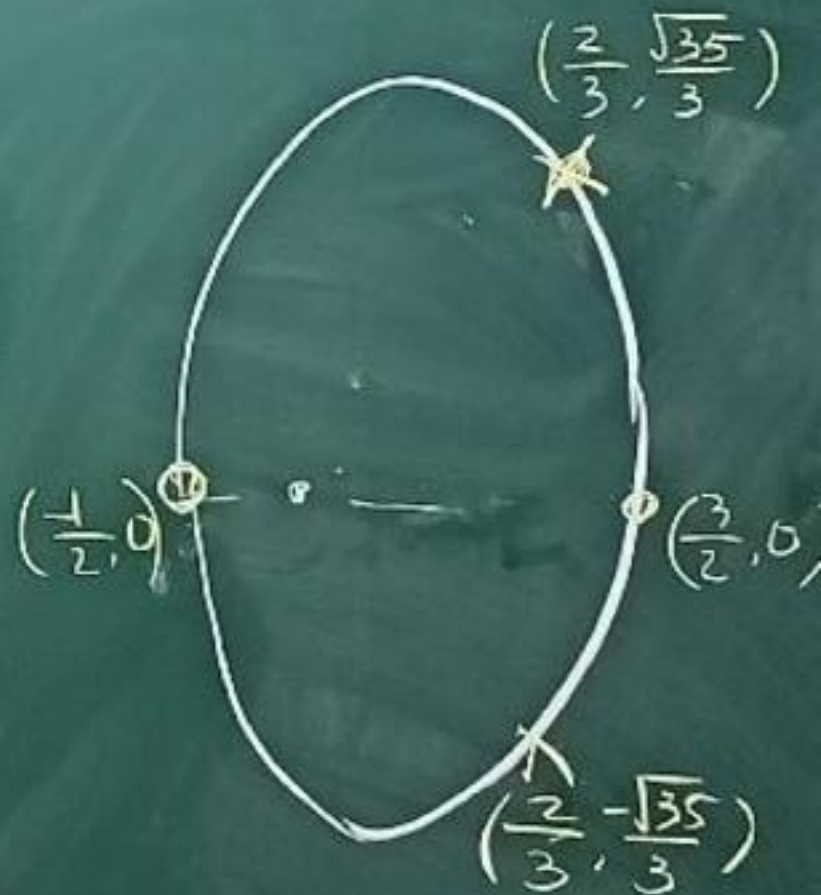
$$(x_0, y_0) = \left(\frac{-1}{2}, 0 \right), \left(\frac{3}{2}, 0 \right), \left(\frac{2}{3}, \frac{\sqrt{35}}{3} \right), \left(\frac{2}{3}, -\frac{\sqrt{35}}{3} \right)$$

$f(x_0, y_0)$ (= distance²)

$$= \frac{1}{4}, \frac{9}{4}, \frac{13}{3}, \frac{13}{3}$$

abs min

abs max



Recall: Find local extremes of $f(x, y)$, subject to $g(x, y) = 0$

\Rightarrow Solve for $(x_0, y_0), \lambda$ from

$$\begin{cases} g(x_0, y_0) = 0 \end{cases}$$

$$\begin{cases} \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$$

3 equations, 3 unknowns

\S Find local extremes of $f(x, y, z)$
subject to the constraint $g(x, y, z) = 0$

\Rightarrow Solve for $(x_0, y_0, z_0), \lambda$ from

$$\begin{cases} g(x_0, y_0, z_0) = 0 \end{cases}$$

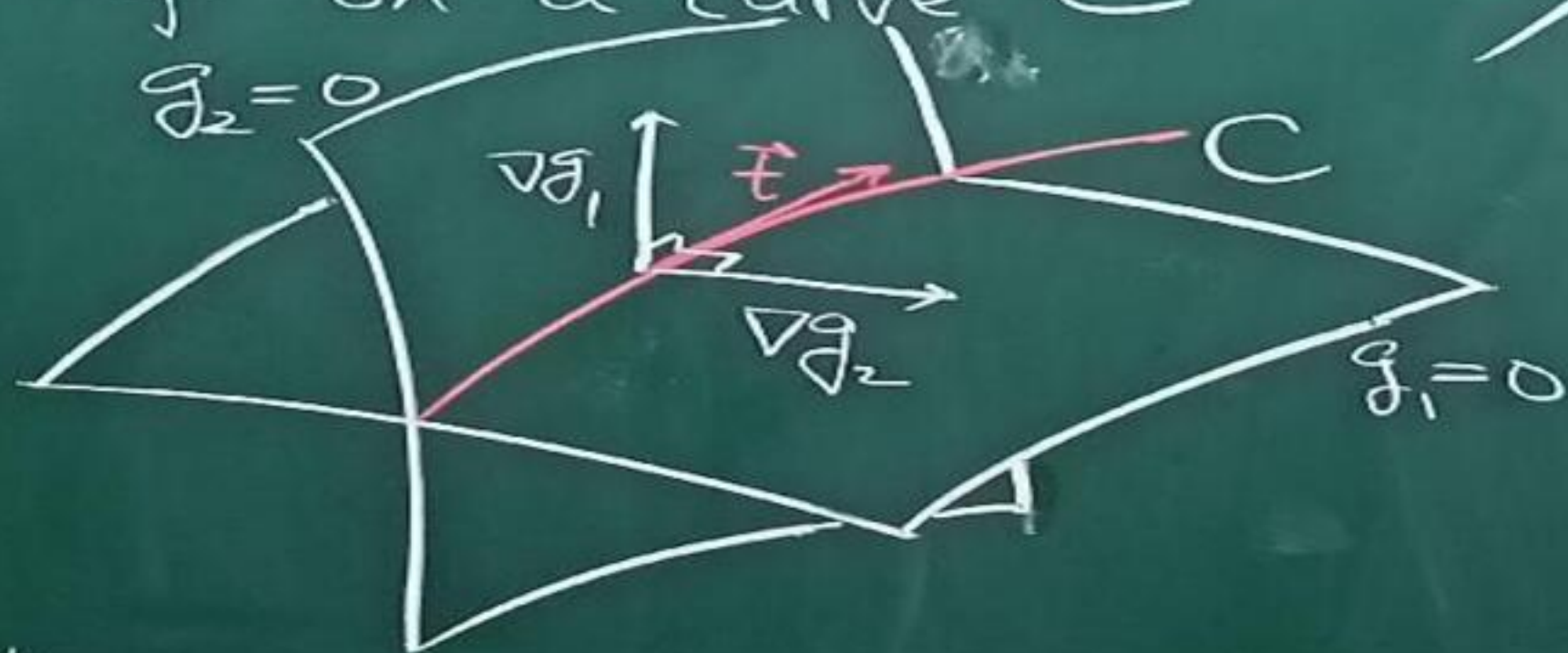
$$\begin{cases} \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \end{cases}$$

4 equations, 4 unknowns

↳ Find local extremes of $f(x, y, z)$ subject to

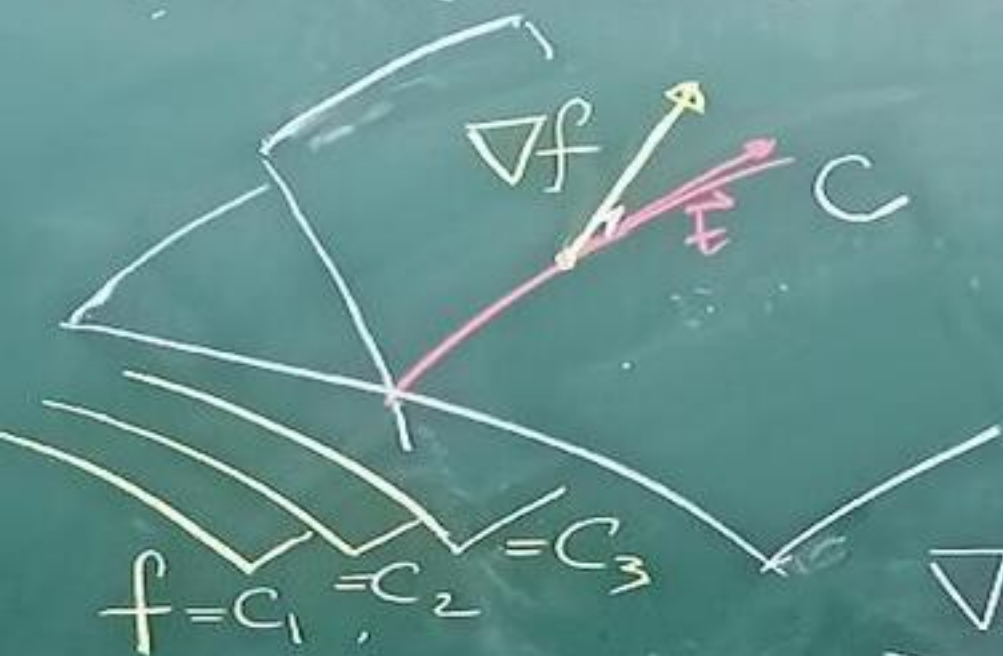
$$\begin{cases} g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases}$$

(Find local extremes of f on a curve C)



(Figure 14.38)

At a local extreme point



\vec{T} : tangent vector
of C at (x_0, y_0, z_0)

$$C \subseteq \{g_1 = 0\}$$

$$\nabla g_1(x_0, y_0, z_0) \perp \{g_1 = 0\}$$

$$\Rightarrow \nabla g_1(x_0, y_0, z_0) \perp \vec{T}$$

Similarly $\nabla g_2(x_0, y_0, z_0) \perp \vec{T}$

Moreover, at a local extreme point

C is tangent to $\{f(x, y, z) = c\}$

$$\Rightarrow \vec{T} \perp \nabla f, \nabla g_1, \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\Rightarrow \nabla f, \nabla g_1, \nabla g_2 \text{ are coplane at } (x_0, y_0, z_0)$$

$$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

Summary:

Solve (x_0, y_0, z_0) , λ_1, λ_2 from

$$g_1(x_0, y_0, z_0) = 0$$

$$g_2(x_0, y_0, z_0) = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\left(\begin{array}{l} \partial_x f = \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2 \\ \partial_y f = \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2 \\ \partial_z f = \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2 \end{array} \right)$$

5 equations, 5 unknowns

Eg 1 Find nearest point

to origin on the curve $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

Solve: Minimize $f(x, y, z) = x^2 + y^2 + z^2$

subject to $\begin{cases} g_1 = x+y+z=1 \\ g_2 = x^2+y^2=1 \end{cases}$

$$x+y+z=1 \quad \text{--- (1)}$$

$$x^2+y^2=1 \quad \text{--- (2)}$$

$$2x = \lambda_1 \cdot 1 + \lambda_2 \cdot 2x \quad \text{--- (3)}$$

$$2y = \lambda_1 \cdot 1 + \lambda_2 \cdot 2y \quad \text{--- (4)}$$

$$2z = \lambda_1 \cdot 1 \quad \text{--- (5)}$$

$$\textcircled{5} \rightarrow \textcircled{3}, \textcircled{4}$$

$$\Rightarrow x(1-\lambda_2) = z$$

$$y(1-\lambda_2) = z$$

$$\Rightarrow \textcircled{a} \lambda_2 = 1, z = 0$$

$$\text{or } \textcircled{b} \lambda_2 \neq 1, x = y$$

Case (a)

$$\textcircled{1} \textcircled{2} \Rightarrow \begin{cases} x+y=1 \\ x^2+y^2=1 \end{cases}$$

$$\Rightarrow (x, y, z) = \begin{matrix} (1, 0, 0) \\ (0, 1, 0) \end{matrix}$$

Case (b) ($x=y$)

$$\textcircled{2} \Rightarrow (x, y) = \pm \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$z = 1 - x - y = 1 \mp \sqrt{2}$$

$$\underline{(1, 0, 0), (0, 1, 0)}, \underline{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2} \right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1+\sqrt{2} \right)}$$

$$f = \underbrace{1, 1}_{\text{abs min}}$$


$$\underline{\underline{4-2\sqrt{2}}},$$

local max

$$\underline{\underline{4+2\sqrt{2}}}$$

abs max

Taylor polynomial of
 $f(x, y)$ centered at (a, b)

$$\begin{aligned}x &= a+h \\ y &= b+k\end{aligned}$$


(a, b) (x, y)

Let $F(t) = f(a+th, b+tk)$
 $h = x-a, k = y-b, t \in \mathbb{R}$

$$F(0) = f(a, b), F(1) = f(x, y)$$

Assume f and all partial derivatives of f are continuous everywhere.

Taylor's Thm for $F(t)$

$$F(t) = P_n(t) + R_n(t)$$

$$\text{(Want } F(1) = P_n(1) + R_n(1)\text{)} \quad (*)$$

$$\text{where } P_n(t) = \sum_{l=0}^n \frac{F^{(l)}(0)}{l!} t^l$$

$$R_n(t) = \frac{F^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

for some c between 0 and t

Take $t=1$

$$P_n(1) = \sum_{l=0}^n \frac{F^{(l)}(0)}{l!}$$

$$R_n(1) = \frac{F^{(n+1)}(c)}{(n+1)!}, \quad 0 < c < 1$$

$$F^{(k)}(0) = ?$$

$$F'(t) = \frac{d}{dt} f(x(t), y(t))$$

$$\text{where } x(t) = a + th, \quad y(t) = b + tk$$

$$= f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

$$= (h\partial_x + k\partial_y) f(x(t), y(t))$$

$$\therefore \frac{d}{dt} F = (h\partial_x + k\partial_y) f$$

$$F''(t) = \frac{d}{dt} F'(t) = \frac{d}{dt} (h\partial_x + k\partial_y) f$$

$$= \partial_x((h\partial_x + k\partial_y) f) \cdot \frac{dx}{dt} + \partial_y((h\partial_x + k\partial_y) f) \cdot \frac{dy}{dt}$$

$$= h\partial_x((h\partial_x + k\partial_y) f) + k\partial_y((h\partial_x + k\partial_y) f)$$

$$= (h\partial_x + k\partial_y)^2 f(x(t), y(t))$$

In Summary

$$F^{(l)}(t) = (h\partial_x + k\partial_y)^l f(a+th, b+tk)$$

for example

$$F^{(3)} = (h^3\partial_x^3 + 3h^2k\partial_x^2\partial_y + 3hk^2\partial_x\partial_y^2 + k^3\partial_y^3) f$$

$$\stackrel{(*)}{\Rightarrow} f(x, y) = \tilde{P}_n(x, y) + \tilde{R}_n(x, y)$$

$$\tilde{P}_n(x, y) = P_n(1) = \sum_{l=0}^n \frac{(h\partial_x + k\partial_y)^l}{l!} f \Big|_{(a, b)}$$

$$\tilde{R}_n(x, y) = R_n(1)$$

$$= \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch, b+ck)$$

$0 < c < 1$

