Eg 1. find local extremes of $f(x,y) = x^2 + y^2 - 4y + 9$

Sol. Method 1

$f_x = 2x$. $f_y = 2(y-2)$

$\Rightarrow$ critical point $= (0,2)$ only

$f_{xx}(0,2) = 2$, $f_{xy}(0,2) = 0$

$f_{yy}(0,2) = 2 \Rightarrow D = -4$

$A > 0$. $D < 0$

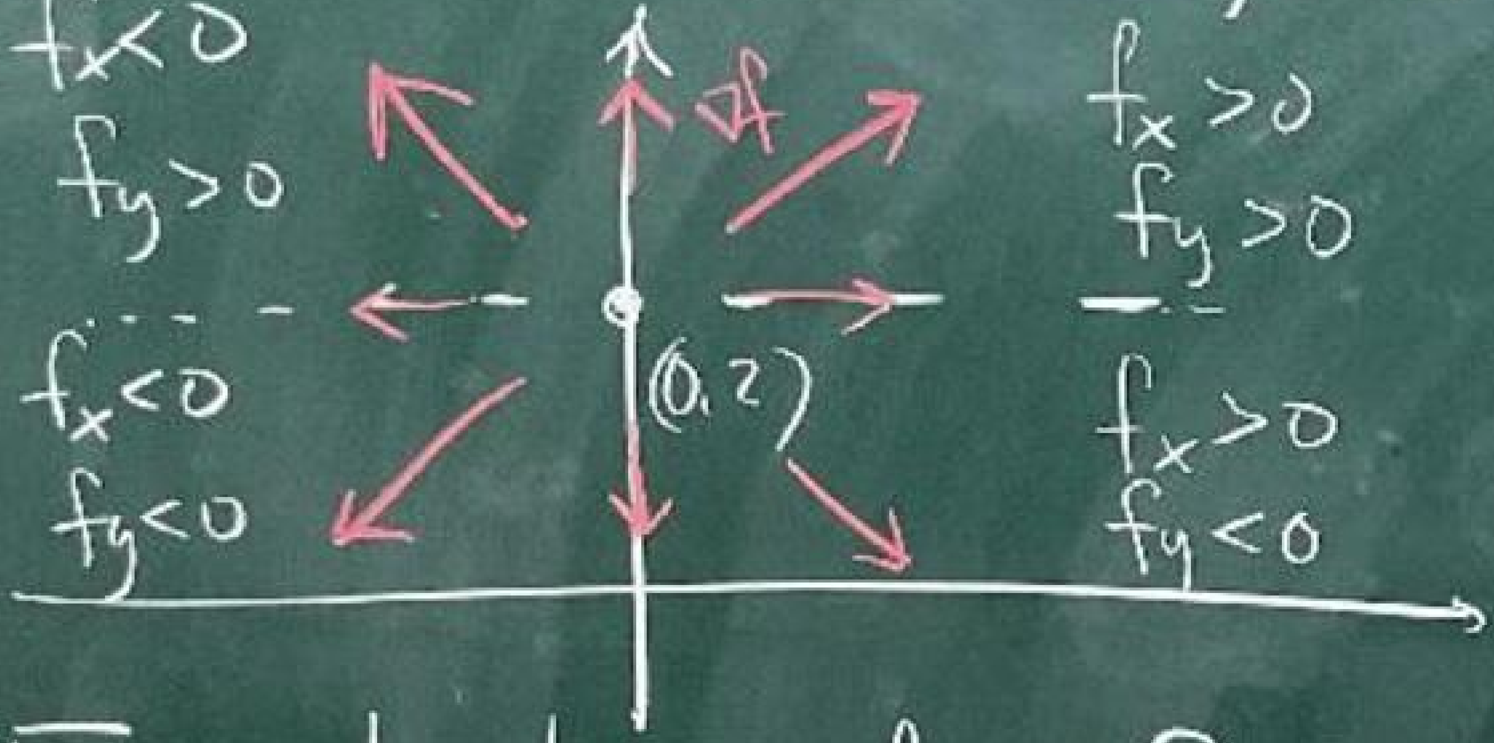
$\Rightarrow$ local min $= f(0,2) = 5$

local max: none

# Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} >0 & \text{if } x>0 \\ <0 & \text{if } x<0 \end{cases}$$

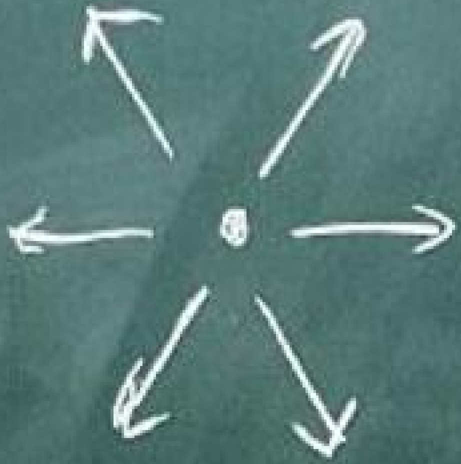
$$f_y = 2(y-2) \begin{cases} >0 & y>2 \\ <0 & y<2 \end{cases}$$




From directions of $\nabla f$ near $(x_0,y_0)=(0,2)$

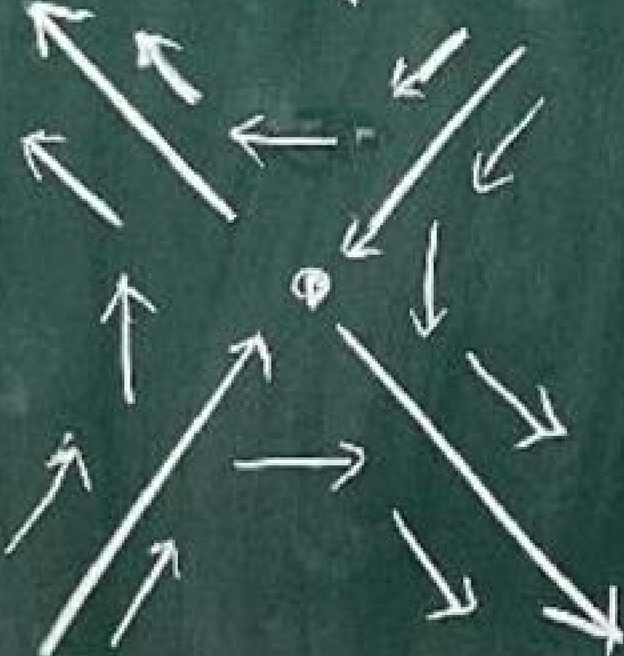
$\Rightarrow f(0,2)$ is a local min

Remark: Near a critical point where $\nabla f(x_0, y_0) = (0,0)$



$\nabla f$ points outward $\Rightarrow$ local min

$\nabla f$ points inward $\Rightarrow$ local max



$\nabla f$ points inward in some directions and outward in some directions $\Rightarrow$ Not local min
Not local max
(e.g. saddle points)

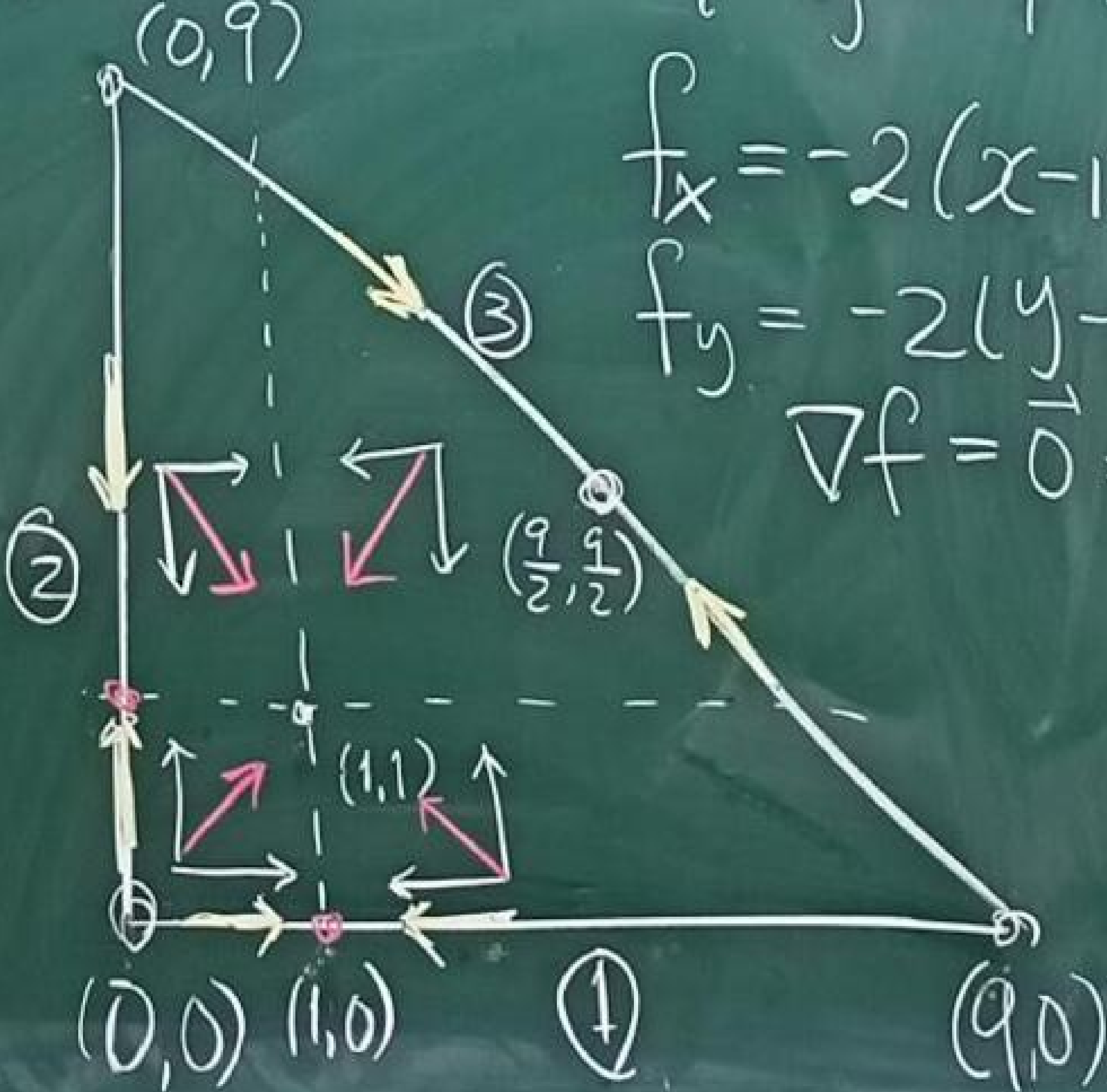
Eg2: Find absolute extremes of $f(x,y)=2+2x+2y-x^2-y^2$ on the region bounded by

$$\begin{cases} x=0 \\ y=0 \\ y=9-x \end{cases}$$

$$f_x=-2(x-1)$$

$$f_y=-2(y-1)$$

$$\nabla f=\vec{0}\Rightarrow(1,1)$$



Method 1 (textbook)

Critical Point = (1.1).

Compare $f(1.1)$ with all values of $f$ on boundary

Method 2 (Gradient Analysis)

Near (1,1)  local max (1,1)

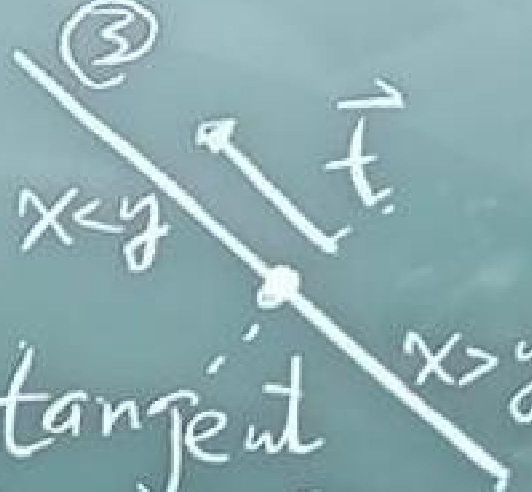
on ① $= \begin{cases} y=0 \\ 0 \le x \le 9 \end{cases}$  local max $= (1,0)$

(1,0) local min $= (0.0), (9,0)$

on ②:  (0,1) local max $= (0,1)$

local min $= (0,0), (0,9)$

On ③ : $\begin{cases} x+y=9 \\ 0 \le x \le 9 \end{cases}$




Let $\vec{t} = (-1, 1)$ = a tangent vector of ③

$\nabla f \cdot \vec{t} \;\begin{matrix} >0 \\ <0 \end{matrix}$ ?

$= (-2(x-1), -2(y-1)) \cdot (-1, 1)$

$= \sqrt{2}(x-y) \begin{cases} >0 \text{ if } x>y \\ <0 \text{ if } x<y \end{cases}$


On ③, local max $= (\frac{9}{2}, \frac{9}{2})$

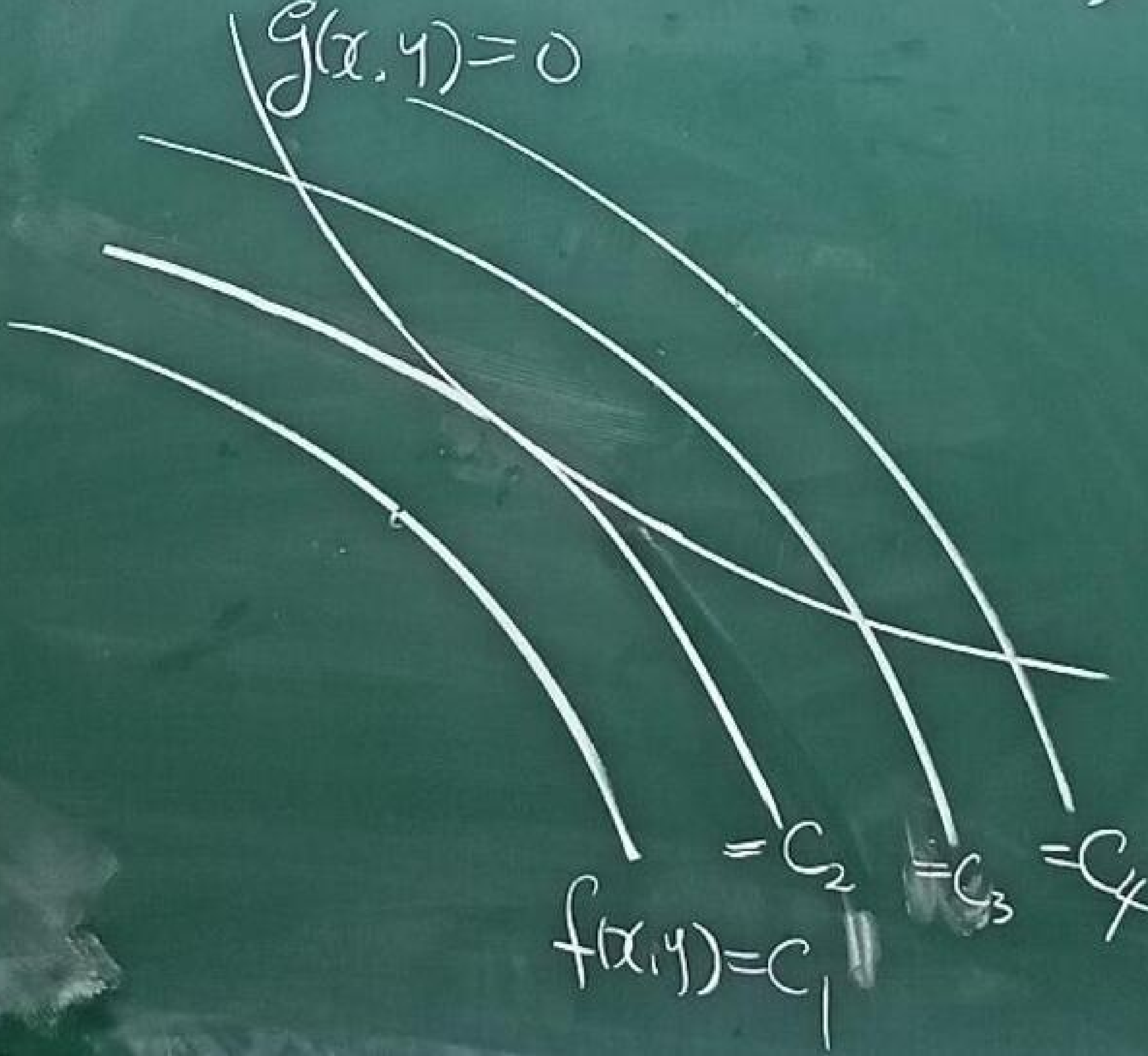
local min $= (9,0), (0,9)$

Absolute max $= f(1,1)$

Abs. min: Compare $f(0,0)$, $f(0,9)$, $f(9,0)$

Constrained optimization
(Lagrangian Multiplier)

<u>Goal</u> Find local min/max of $f(x,y)$ on $\{g(x,y)=0\}$



Local extreme at $(x_0, y_0)$

$\Rightarrow \begin{cases} g(x,y)=0 \\ f(x,y)=c \end{cases}$ are tangent at $(x_0, y_0)$

$\Rightarrow \nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$

$\Rightarrow \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

for some constant $\lambda$

$\Rightarrow$ Solve $(x_0, y_0)$, $\lambda$

from $\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$

3 equations, 3 unknowns

Eg 1: Find the nearest point to origin on $(x-\frac{1}{2})^2+\frac{y^2}{4}=1$

<u>Sol</u>: Minimize $f(x,y)=(x-0)^2+(y-0)^2$

subject to $g(x,y)=(x-\frac{1}{2})^2+\frac{y^2}{4}-1=0$

$$\begin{cases} (x_0-\frac{1}{2})^2+\frac{y_0^2}{4}=1 & (g=0) & (1)\\ 2x_0=\lambda(x_0-\frac{1}{2}) & (f_x=\lambda g_x) & (2)\\ 2y_0=\lambda\cdot\frac{y_0}{2} & (f_y=\lambda g_y) & (3)\end{cases}$$

$(3)\Rightarrow y_0=0$ or $\lambda=4$

<u>Case A</u>: $y_0=0$

$$\begin{cases} (x_0-\frac{1}{2})^2=1 & (A_1)\\ 2x_0=2\lambda(x_0-\frac{1}{2}) & (A_2)\end{cases}$$

$(A_1) \Rightarrow x_0 = -\frac{1}{2}$ or $\frac{3}{2}$

"+"$(A_2) \Rightarrow \lambda = \frac{1}{2}$ $\quad 3$

<u>Case B</u> $\lambda = 4$

"+"(2) $\Rightarrow x_0 = \frac{2}{3}$

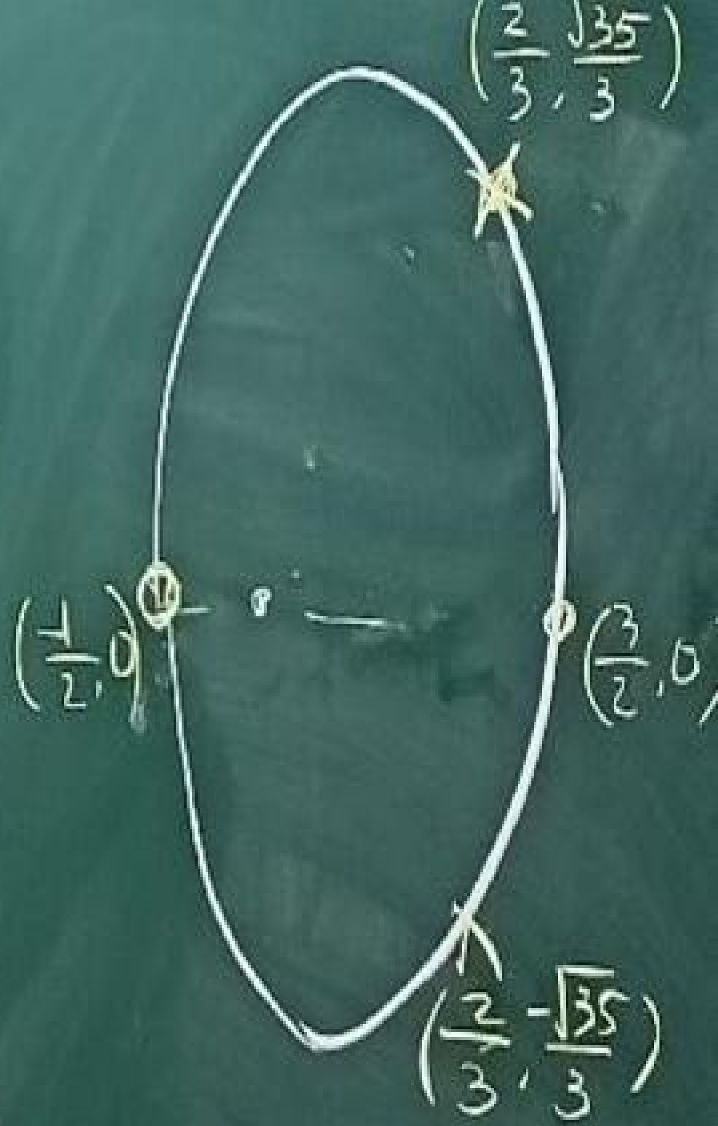
"+"(1) $\Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$

Overall, we have

nearest

$(x_0, y_0) = \underline{\left(-\frac{1}{2}, 0\right)}, \left(\frac{3}{2}, 0\right)$

$\left(\frac{2}{3}, \frac{\sqrt{35}}{3}\right), \left(\frac{2}{3}, \frac{-\sqrt{35}}{3}\right)$




$f(x_0, y_0)$ (= distance$^2$)

$= \underbrace{\frac{1}{4}}_{\text{abs min}}, \frac{9}{4}, \underbrace{\frac{13}{3}, \frac{13}{3}}_{\text{abs max}}$

Recall: Find local extremes of $f(x,y)$ subject to $g(x,y)=0$

$\Rightarrow$ Solve for $(x_0,y_0)$, $\lambda$ from

$$\begin{cases} g(x_0,y_0)=0 \\ \nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0) \end{cases}$$

$\underline{\underline{3}}$ equations, $\underline{\underline{3}}$ unknowns

---

§ Find local extremes of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$

$\Rightarrow$ Solve for $(x_0,y_0,z_0)$, $\lambda$ from

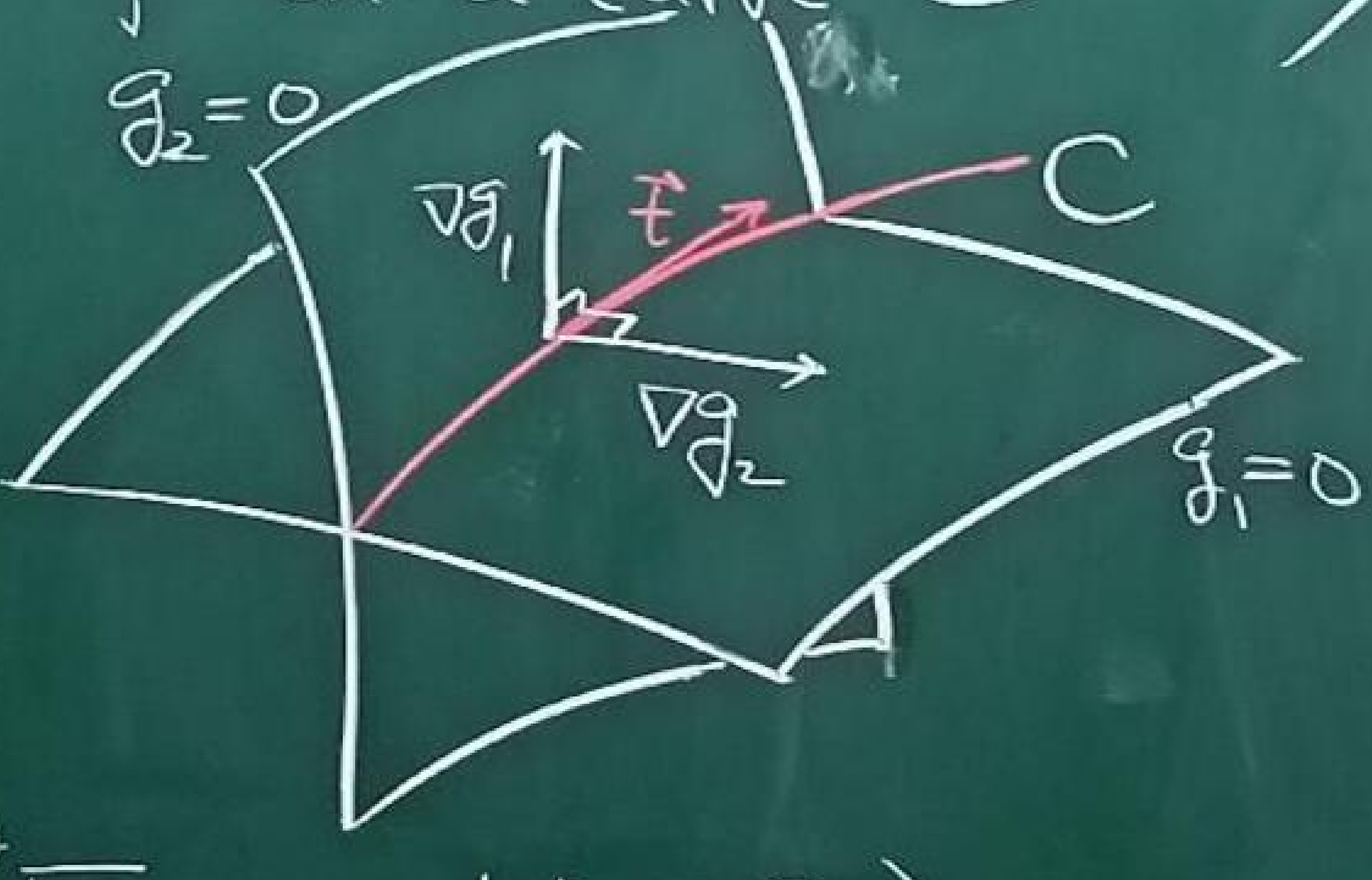
$$\begin{cases} g(x_0,y_0,z_0)=0 \\ \nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0) \end{cases}$$

$\underline{\underline{4}}$ equations, $\underline{\underline{4}}$ unknowns

§ Find local extremes of $f(x,y,z)$ subject to

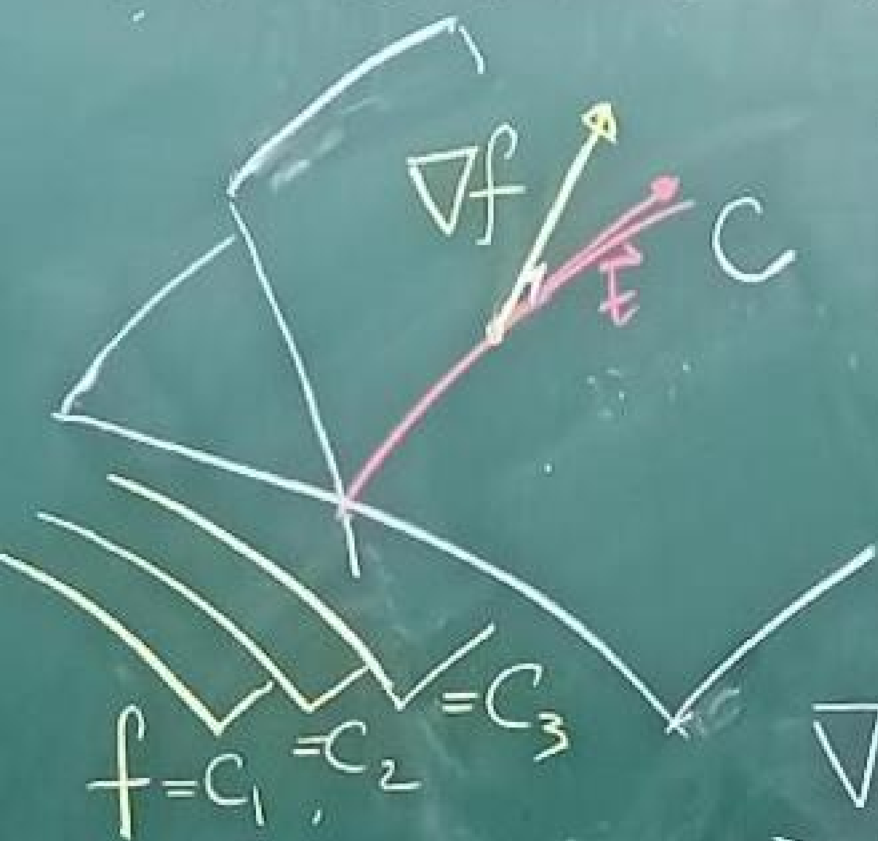
$$\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$

(Find local extremes of $f$ on a curve $C$)




(Figure 14.38)

At a local extreme point




$\vec{t}$ : tangent vector of $C$ at $(x_0, y_0, z_0)$

$C \subseteq \{g_1 = 0\}$

$\nabla g_1(x_0, y_0, z_0) \perp \{g_1 = 0\}$

$\Rightarrow \nabla g_1(x_0, y_0, z_0) \perp \vec{t}$

Similarly $\nabla g_2(x_0, y_0, z_0) \perp \vec{t}$

Moreover, at a local extreme point $C$ is tangent to $\{f(x,y,z) = c\}$

$\Rightarrow \vec{t} \perp \nabla f, \nabla g_1, \nabla g_2$ at $(x_0, y_0, z_0)$

$\Rightarrow \nabla f, \nabla g_1, \nabla g_2$ are co-plane at $(x_0, y_0, z_0)$

$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ at $(x_0, y_0, z_0)$

Summary:

Solve $(x_0, y_0, z_0)$, $\lambda_1$, $\lambda_2$ from

$$g_1(x_0, y_0, z_0) = 0$$

$$g_2(x_0, y_0, z_0) = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\left( \begin{aligned} \partial_x f &= \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2 \\ \partial_y f &= \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2 \\ \partial_z f &= \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2 \end{aligned} \right)$$

$\underline{\underline{5}}$ equations, $\underline{\underline{5}}$ unknowns

Eg1 Find nearest point to origin on the curve $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

<u>Solve</u>: Minimize $f(x,y,z)=x^2+y^2+z^2$

subject to $\begin{cases} g_1 = x+y+z=1 \\ g_2 = x^2+y^2=1 \end{cases}$

$$x+y+z=1 \quad \text{— ①}$$

$$x^2+y^2 \quad =1 \quad \text{— ②}$$

$$2x = \lambda_1 1 + \lambda_2 \cdot 2x \quad \text{— ③}$$

$$2y = \lambda_1 1 + \lambda_2 \cdot 2y \quad \text{— ④}$$

$$2z = \lambda_1 \cdot 1 \quad \text{— ⑤}$$

(5) $\rightarrow$ (3), (4)

$\Rightarrow x(1-\lambda_2) = z$

$y(1-\lambda_2) = z$

$\Rightarrow$ (a) $\lambda_2 = 1,\ z = 0$

or (b) $\lambda_2 \neq 1,\ x = y$

Case (a)

(1)(2) $\Rightarrow \begin{cases} x+y=1 \\ x^2+y^2=1 \end{cases}$

$\Rightarrow (x,y,z) = \boxed{\begin{matrix}(1,0,0)\\(0,1,0)\end{matrix}}$

Case (b) $(x=y)$

(2) $\Rightarrow (x,y) = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$z = 1-x-y = 1 \mp \sqrt{2}$

$(1,0,0), (0,1,0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$

$f = \underbrace{1,\ 1}_{\text{abs min}}\quad \underbrace{4-2\sqrt{2}}_{\text{local max}},\quad \underbrace{4+2\sqrt{2}}_{\text{abs max}}$

Taylor polynomial of $f(x,y)$ centered at $(a,b)$

$x=a+h$
$y=b+k$


Let $F(t)=f(a+th,\ b+tk)$

$h=x-a,\ k=y-b,\ t\in\mathbb{R}$

$F(0)=f(a,b),\ F(1)=f(x,y)$

Assume $f$ and all partial derivatives of $f$ are continuous everywhere.

# Taylor's Thm for $F(t)$

$$F(t) = P_n(t) + R_n(t)$$

(Want $F(1) = P_n(1) + R_n(1)$.) (*)

where $P_n(t) = \sum_{l=0}^{n} \frac{F^{(l)}(0)}{l!} t^l$

$$R_n(t) = \frac{F^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

for some $c$ between $0$ and $t$

Take $t = 1$

$$P_n(1) = \sum_{l=0}^{n} \frac{F^{(l)}(0)}{l!}$$

$$R_n(1) = \frac{F^{(n+1)}(c)}{(n+1)!}, \quad 0 < c < 1$$

$$F^{(k)}(0) = ?$$

$$F'(t) = \frac{d}{dt} f(x(t), y(t))$$

where $x(t) = a + th$, $y(t) = b + tk$

$$= f_x(x(t), y(t)) \cdot \underbrace{x'(t)}_{= h} + f_y(x(t), y(t)) \cdot \underbrace{y'(t)}_{= k}$$

$$= (h\partial_x + k\partial_y) f(x(t), y(t))$$

$$\therefore \underline{\frac{d}{dt} F = (h\partial_x + k\partial_y) f}$$

$$F''(t) = \frac{d}{dt} F'(t) = \frac{d}{dt} \overbrace{(h\partial_x + k\partial_y) f}^{f_1(x(t), y(t))}$$

$$= \partial_x(\underbrace{(h\partial_x + k\partial_y) f}_{f_1}) \cdot \frac{dx}{dt} + \partial_y((h\partial_x + k\partial_y) f) \cdot \frac{dy}{dt}$$

$$= h\partial_x((h\partial_x + k\partial_y) f) + k\partial_y((h\partial_x + k\partial_y) f)$$

$$= (h\partial_x + k\partial_y)^2 f(x(t), y(t))$$

In Summary

$$F^{(\ell)}(t) = (h\partial_x + k\partial_y)^{\ell} f(a+th, b+tk)$$

for example:

$$F^{(3)} = \left(h^3\partial_x^3 + 3h^2k\,\partial_x^2\partial_y + 3hk^2\,\partial_x\partial_y^2 + k^3\partial_y^3\right) f$$

$$\xRightarrow{(*)} f(x,y) = \tilde{P}_n(x,y) + \tilde{R}_n(x,y)$$

$$\tilde{P}_n(x,y) = P_n(1) = \sum_{\ell=0}^{n} \frac{(h\partial_x + k\partial_y)^{\ell}}{\ell!} f \Big|_{(a,b)}$$

$$\tilde{R}_n(x,y) = R_n(1)$$

$$= \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch, b+ck) \qquad 0<c<1$$