Eg 1. Find local extremes of $f(x,y) = x^2 + y^2 - 4y + 9$

Sol. Method 1

$f_x = 2x$, $f_y = 2(y-2)$

$\Rightarrow$ critical point $= (0,2)$ only

$f_{xx}(0,2) = 2$, $f_{xy}(0,2) = 0$

$f_{yy}(0,2) = 2 \Rightarrow D = -4$

$A > 0$, $D < 0$

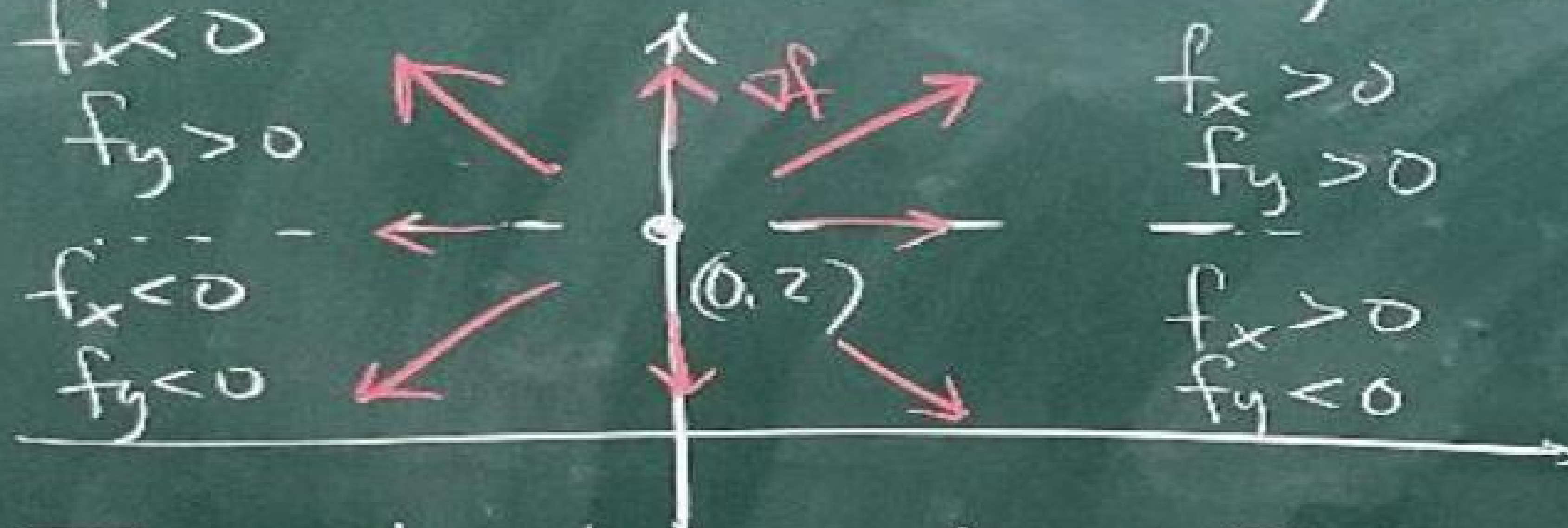
$\Rightarrow$ local min $= f(0,2) = 5$

local max: none

Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} >0 & \text{if } x>0 \\ <0 & \text{if } x<0 \end{cases}$$

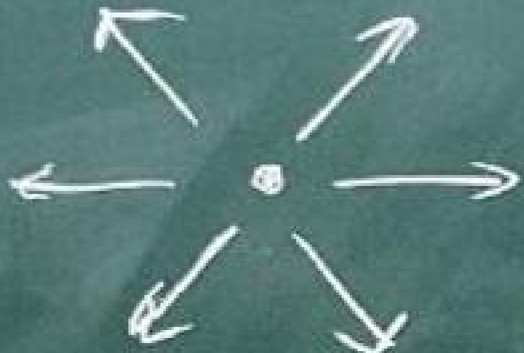
$$f_y = 2(y-2) \begin{cases} >0 & y>2 \\ <0 & y<2 \end{cases}$$



From directions of ∇f near $(x_0, y_0) = (0,2)$

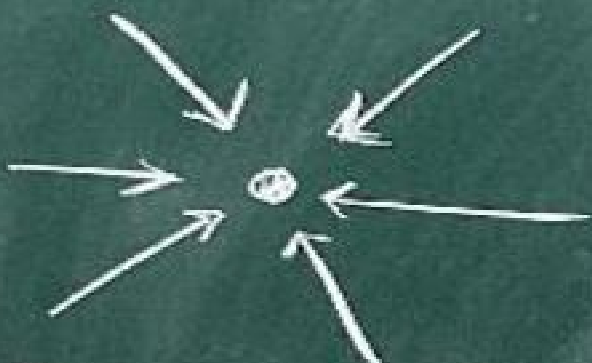
$\Rightarrow f(0,2)$ is a local min

Remark: Near a critical point where $\nabla f(x_0, y_0) = (0,0)$



∇f points outward

$\Rightarrow$ local min



∇f points inward

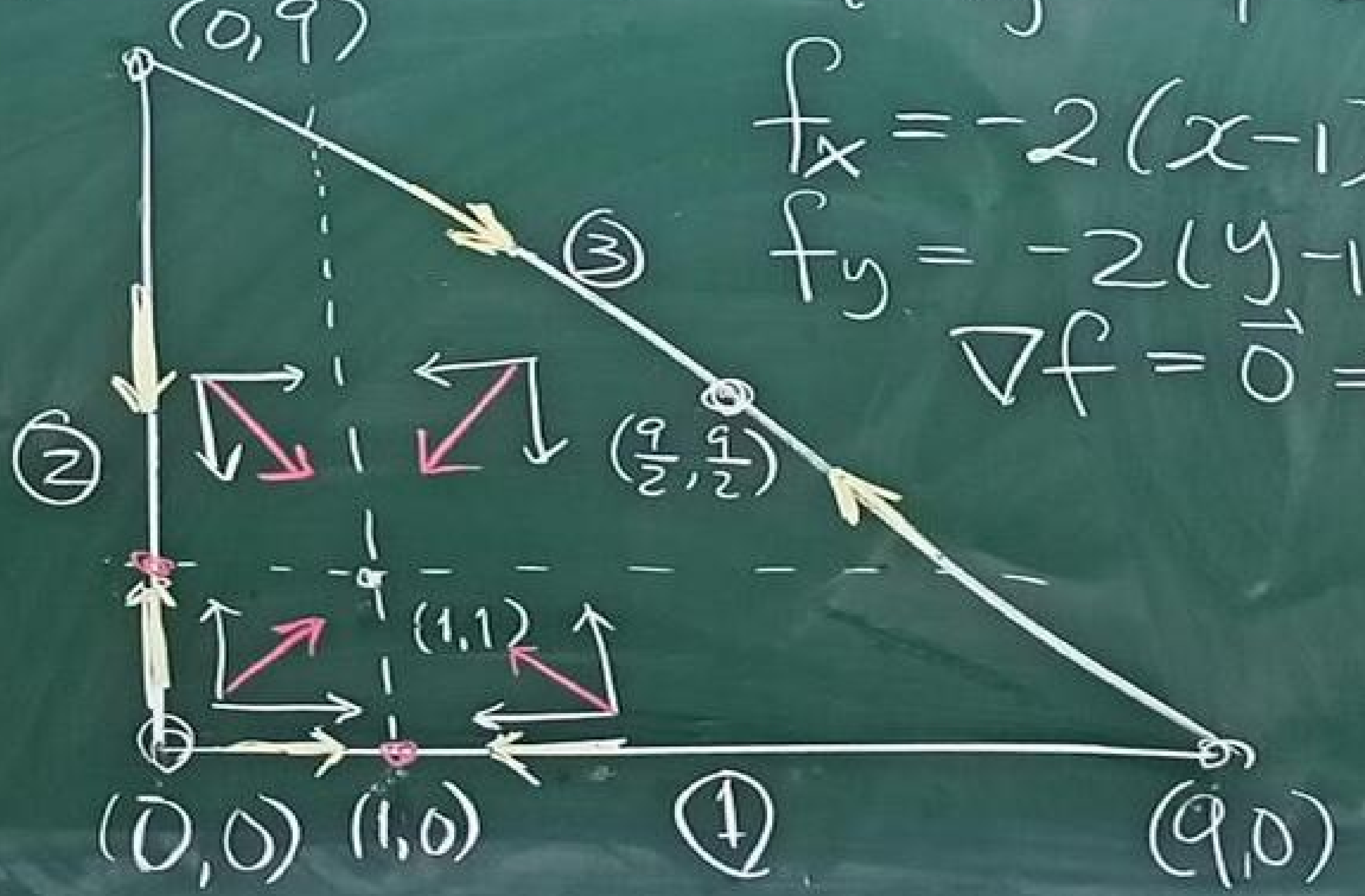
$\Rightarrow$ local max

∇f points inward in some directions and outward in some directions $\Rightarrow$ Not local min

Not local max

(e.g. saddle points)

Eg2: Find absolute extremes of $f(x,y)=2+2x+2y-x^2-y^2$ on the region bounded by
$$\begin{cases} x=0 \\ y=0 \\ y=9-x \end{cases}$$



$$f_x=-2(x-1)$$
$$f_y=-2(y-1)$$
$$\nabla f=\vec{0}\Rightarrow(1,1)$$

Method 1 (textbook)

Critical Point = (1.1).

Compare f(1.1) with all
values of f on boundary

Method 2 (Gradient Analysis)

Near (1,1)

local max (1,1)

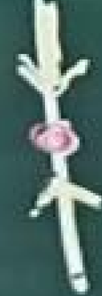
on ① = $\begin{cases} y=0 \\ 0 \le x \le 9 \end{cases}$

(1,0)

local max = (1,0)

local min = (0.0), (9.0)

on ②:



(0,1) local max = (0,1)

local min = (0,0), (0,9)

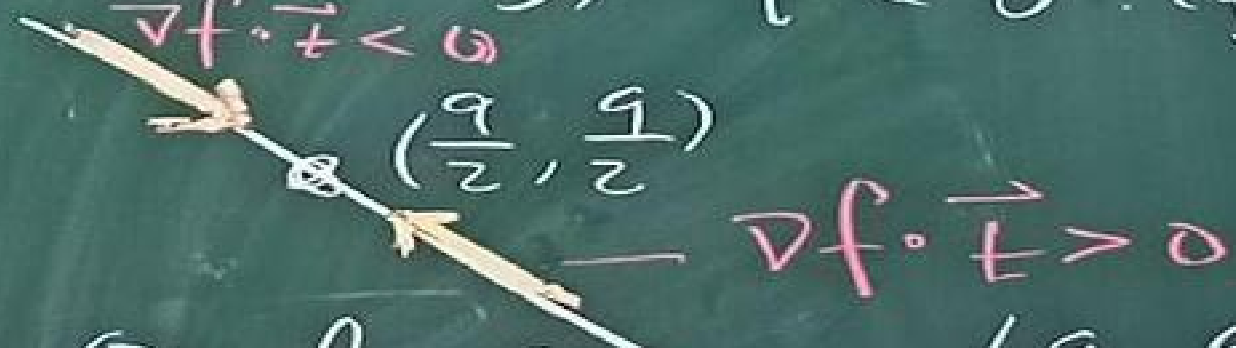
On ③ $= \begin{cases} x+y=9 \\ 0 \le x \le 9 \end{cases}$

Let $\vec{t} = (-1, 1)$ = a tangent vector of ③

$\nabla f \cdot \vec{t} \begin{matrix} >0 \\ <0 \end{matrix}$?

$= (-2(x-1), -2(y-1)) \cdot (-1, 1)$

$= \sqrt{2}(x-y) \begin{cases} >0 & \text{if } x>y \\ <0 & \text{if } x<y \end{cases}$



On ③, local max $= (\frac{9}{2}, \frac{9}{2})$

local min $= (9,0), (0,9)$

Absolute max $= f(1,1)$

Abs. mini: Compare $f(0,0)$, $f(0,9)$, $f(9,0)$