

Ex 1. find local extremes

$$\text{of } f(x, y) = x^2 + y^2 - 4y + 9$$

Sol. Method 1

$$f_x = 2x, \quad f_y = 2(y - 2)$$

\Rightarrow critical point = $(0, 2)$ only

$$f_{xx}(0, 2) = 2, \quad f_{xy}(0, 2) = 0$$

$$f_{yy}(0, 2) = 2 \Rightarrow D = -4$$

$$A > 0, \quad D < 0$$

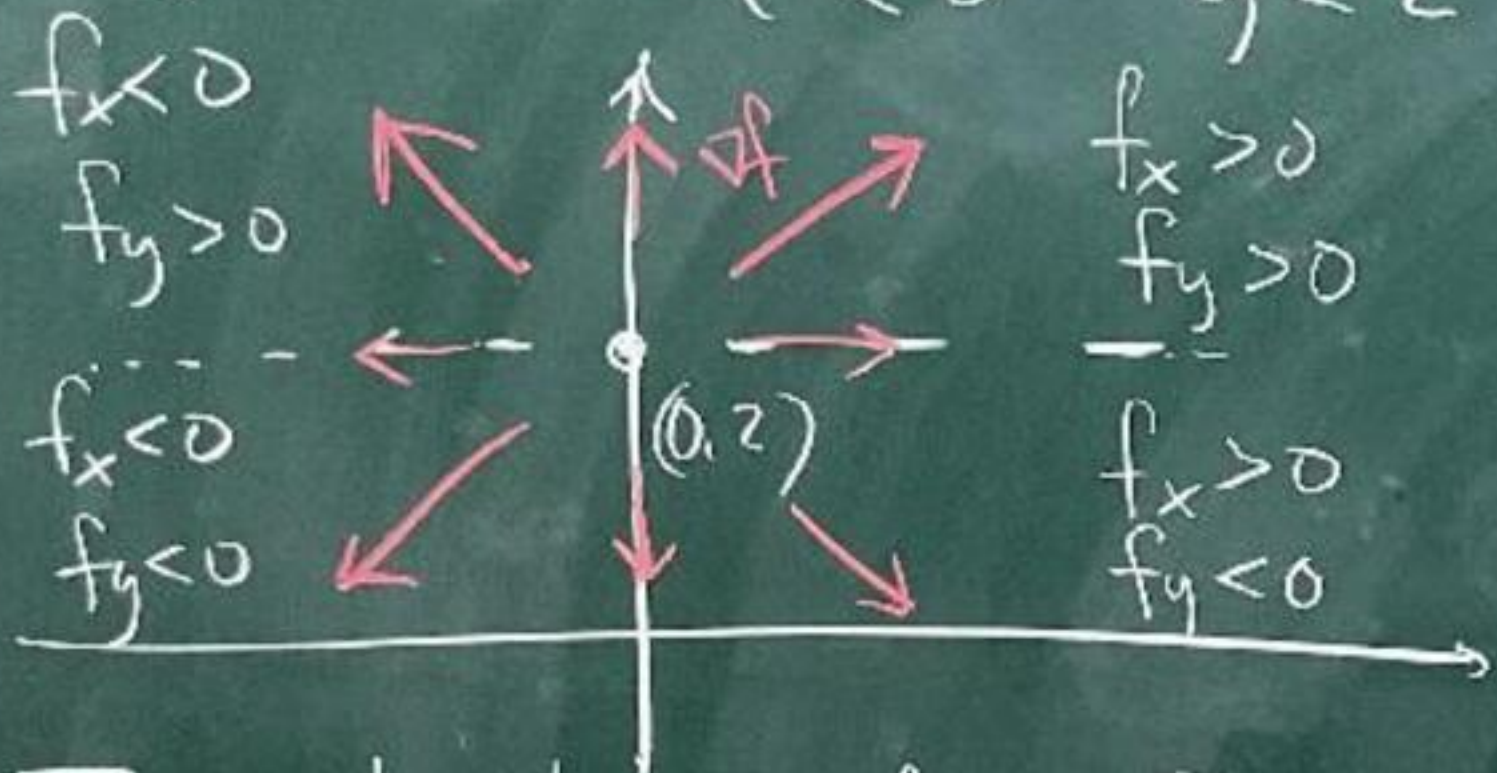
$$\Rightarrow \text{local min} = f(0, 2) = 5$$

local max: none

Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

$$f_y = 2(y-2) \begin{cases} > 0 & \text{if } y > 2 \\ < 0 & \text{if } y < 2 \end{cases}$$



From directions of ∇f near
 $(x_0, y_0) = (0, 2)$

$\Rightarrow f(0, 2)$ is a local min

Remark: Near a critical point where $\nabla f(x_0, y_0) = (0, 0)$



∇f points outward
 \Rightarrow local min



∇f points inward
 \Rightarrow local max



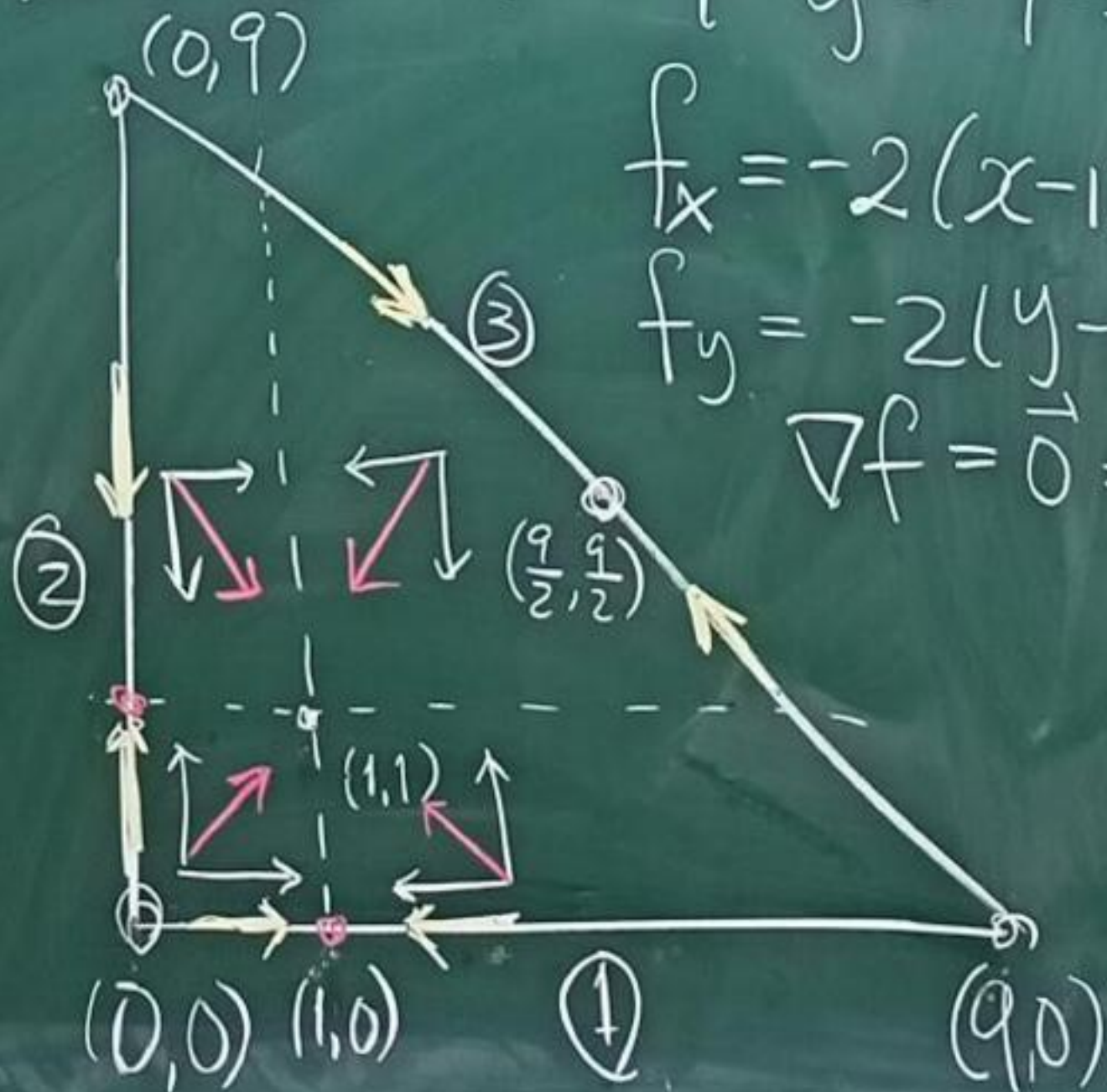
∇f points inward in some directions and outward in some directions \Rightarrow Not local min
Not local max
(eg. saddle points)

Ex 2: Find absolute extremes

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the region
bounded by

$$\begin{cases} x=0 \\ y=0 \\ y=9-x \end{cases}$$



$$f_x = -2(x-1)$$

$$f_y = -2(y-1)$$

$$\nabla f = \vec{0} \Rightarrow (1,1)$$

Method 1 (textbook)

Critical Point = (1, 1)

Compare $f(1, 1)$ with all values of f on boundary

Method 2 (Gradient Analysis)

Near (1, 1)



local max (1, 1)

on ① = $\begin{cases} y=0 \\ 0 \leq x \leq 9 \end{cases}$



on ②:



(0, 1)

local max = (0, 1)

local min = (0, 0), (0, 9)

On ③ = $\begin{cases} x+y=9 \\ 0 \leq x \leq 9 \end{cases}$



Let $\vec{t} = (-1, 1)$ = a tangent vector of ③

$$\nabla f \cdot \vec{t} \geq 0 ?$$

$$\nabla f \cdot \vec{t} \leq 0 ?$$

$$= (-2(x-1), +2(y-1)) \cdot (-1, 1)$$

$$= \sqrt{2}(x-y) \begin{cases} > 0 & \text{if } x > y \\ < 0 & \text{if } x < y \end{cases}$$



On ③, local max = $(\frac{9}{2}, \frac{9}{2})$

local min = $(9, 0), (0, 9)$

Absolute max = $f(1, 1)$

Abs. min: Compare $f(0, 0), f(0, 9), f(9, 0)$