

Ex 2. Find the tangent plane and normal line of  $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$  at  $(1, 2, 3)$ .

Sol tangent plane  $\perp \nabla F(1, 2, 3)$   
normal line  $\parallel \nabla F(1, 2, 3)$

$$F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$$

$$\text{Surface: } F(x, y, z) = 3$$

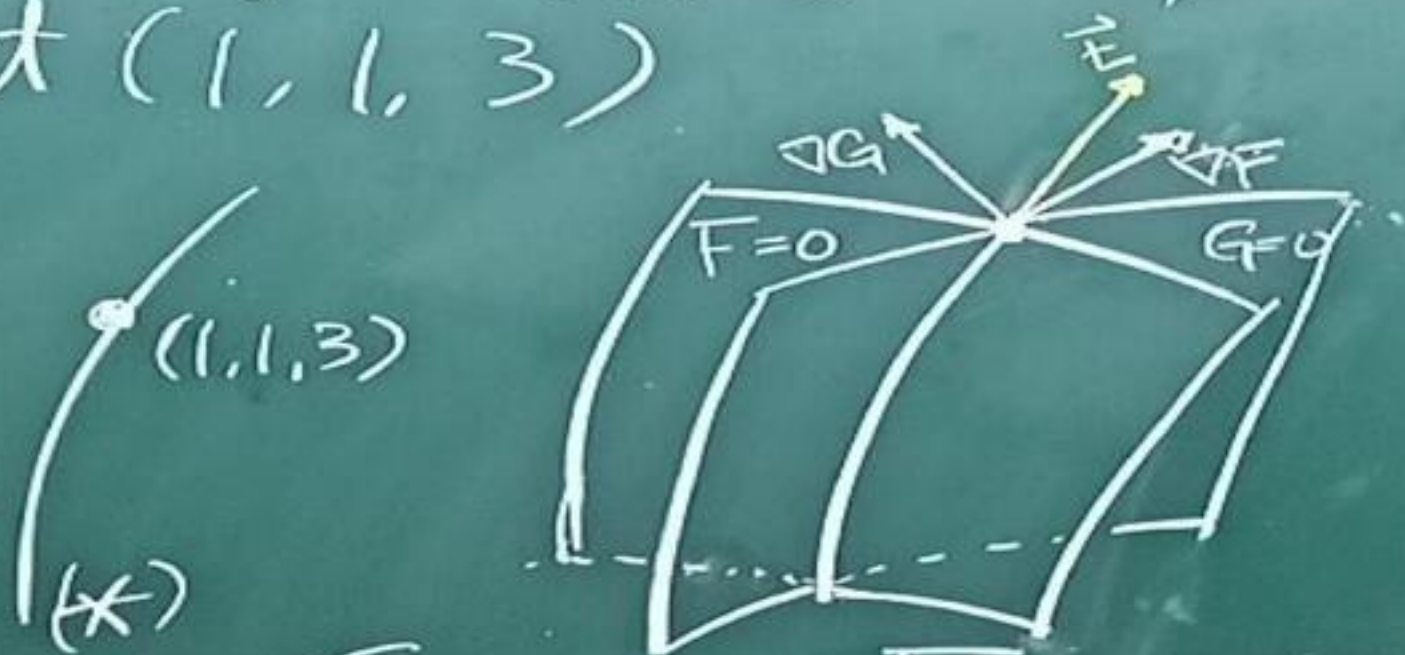
$$\nabla F = (F_x, F_y, F_z) = \left(2x, \frac{y}{2}, \frac{2z}{9}\right)$$

$$\nabla F(1, 2, 3) = \left(2, 1, \frac{2}{3}\right)$$

$$\text{tangent plane: } (x-1, y-2, z-3) \cdot \left(2, 1, \frac{2}{3}\right) = 0$$

$$\text{normal line } \begin{cases} x(t) = 1 + 2t \\ y(t) = 2 + t \\ z(t) = 3 + \frac{2}{3}t \end{cases}$$

Eg3 Find the tangent  
 line of  $\begin{cases} x^2 + y^2 - z = 0 \\ x + z = 4 \end{cases} (*)$   
 at  $(1, 1, 3)$



If  $C = \left\{ (x, y, z), \begin{matrix} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{matrix} \right\}$   
 and  $(x_0, y_0, z_0) \in C$

$\vec{t}$ : tangent vector of  $C$   
 at  $(x_0, y_0, z_0)$ . Here  $F(x, y, z) = x^2 + y^2 - z$   
 $G(x, y, z) = x + z - 4$

$$C \subset S_1 =$$

Since  $C \subset S_1 = \left\{ (x, y, z), F(x, y, z) = \text{const} \right. \\ \left. = F(x_0, y_0, z_0) \right\}$

$\Rightarrow \vec{T}$  is a tangent vector of  $S_1$  at  $(x_0, y_0, z_0)$

$$\Rightarrow \vec{T} \perp \nabla F(x_0, y_0, z_0)$$

Similarly,  $\vec{T} \perp \nabla G(x_0, y_0, z_0)$

$$\therefore \vec{T} \parallel \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$$

$$= (2x_0, 2y_0, 0) \times (1, 0, 1)$$

$$= (2, 2, 0) \times (1, 0, 1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2, -2, -2)$$

$$\text{Ans } \frac{x-1}{2} = \frac{y-2}{-2} = \frac{z-3}{-2}$$

$$\text{or } x(t) = 1 + 2t, \quad y(t) = 2 - 2t, \quad z(t) = 3 - 2t$$

# Linearization

If  $f(x, y)$  is diff. at  $(x_0, y_0)$

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$\therefore L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$   
is a good approximation of  $z = f(x, y)$   
near  $(x_0, y_0, f(x_0, y_0))$

Linearization: use  $L(x, y)$  to approx.  $f(x, y)$

$$\Leftrightarrow \Delta z \cong f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \quad (\star)$$
  
( $df = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$ )

Note: If "=" holds in  $(\star)$

$$\Rightarrow z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow z = L(x, y)$$

Eg 1:  $f(x, y) = x^2 - xy + \frac{y^2}{2} + 3$

Find approx. value of  $f(3.1, 1.9)$

Sol: take  $(x_0, y_0) = (3, 2)$

$$f(3, 2) = 8, \quad f_x(3, 2) = 4$$

$$f_y(3, 2) = -1$$

App. value =  $L(3.1, 1.9)$

$$= f(3, 2) + f_x(3, 2) \cdot (3.1 - 3) + f_y(3, 2) \cdot (1.9 - 2)$$

$$= 8 + 4 \cdot 0.1 + (-1) \cdot (-0.1) = 8.5$$

Remark: The same as

$$\Delta z = z - f(3, 2) \approx f_x(3, 2) \Delta x + f_y(3, 2) \Delta y$$

$$\Rightarrow z \approx 8.5$$

# Error of linearization

$$f(x, y) \cong L(x, y)$$

How large is  $|f(x, y) - L(x, y)|$ ?

1D case ( $y = f(x)$ )

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{L(x)} + \underbrace{\frac{f''(\xi)}{2}(x - x_0)^2}_{\text{error}}$$

(Taylor's formula ( $n=1$ ) on P 883)

2D case ( $z = f(x, y)$ )

$$f(x, y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}$$

$$+ \frac{1}{2} \left( f_{xx}(\xi_1, \xi_2)(x - x_0)^2 + 2f_{xy}(\xi_1, \xi_2)(x - x_0)(y - y_0) + f_{yy}(\xi_1, \xi_2)(y - y_0)^2 \right)$$

where  $(\xi_1, \xi_2)$  lies on the segment  $(x_0, y_0) (x, y)$  error

Corollary: If  $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$  are all continuous in an open region  $R$ ,  $(x_0, y_0) \in R$

and  $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$  on  $R$

Then  $|f(x, y) - L(x, y)|$

$$\leq \frac{1}{2} (M|x-x_0|^2 + 2M|x-x_0||y-y_0| + M|y-y_0|^2)$$

$$= \frac{M}{2} (|x-x_0| + |y-y_0|)^2$$

Error estimate for linear approximation

---

In Eg 1,  $f_{xx} = 2, f_{xy} = -1, f_{yy} = 1$  ( $\therefore M = 2$ )

$$\therefore |f(3.1, 1.9) - 8.5| \leq \frac{M}{2} (0.1 + 0.1)^2 = 0.04$$

# Eg 2: Volume of Cylinder

$$V(r, h) = \pi r^2 h$$

If  $r_0 = 1, h_0 = 5$

$$r_1 = r_0 + \Delta r, \quad \Delta r = 0.1$$

$$h_1 = h_0 + \Delta h, \quad \Delta h = -0.1$$

$$\left[ \begin{array}{l} \therefore \\ |\text{Error}| \\ \leq \frac{10\pi}{2} (0.1 + 0.1)^2 \\ = 0.2\pi \end{array} \right]$$

$$\Delta V = V(r_1, h_1) - V(r_0, h_0) = ?$$

Ans:  $\Delta V \approx V_r(r_0, h_0) \Delta r + V_h(r_0, h_0) \Delta h$

$$= 2\pi r_0 h_0 \Delta r + \pi r_0^2 \Delta h = 0.9\pi$$

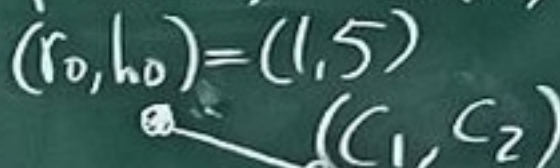
$$|\text{Error}| \leq \frac{M}{2} (|\Delta r| + |\Delta h|)^2$$

Find  $M$ , such that  $|V_{rr}|, |V_{rh}|, |V_{hh}| \leq M$

$$V_{rr}(C_1, C_2) = 2\pi C_2 \quad (r_0, h_0) = (1, 5)$$

$$V_{rh}(C_1, C_2) = 2\pi C_1$$

$$|C_1| \leq 1.1, |C_2| \leq 5 \Rightarrow M = 10\pi \text{ will do}$$



$(r_1, h_1) = (1.1, 4.9)$

# Extreme Values and Saddle points

How to find local min/max of  $f(x, y)$   
(assuming all partial derivatives are cont.)

(1) First derivative test

$f$  has local min/max at  $(x_0, y_0)$

$$\Leftrightarrow \nabla f(x_0, y_0) = (0, 0) \quad \text{leading term}$$

$$\therefore \Delta z = f(x, y) - f(x_0, y_0) = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y) + \underbrace{(\varepsilon_1, \varepsilon_2) \cdot (\Delta x, \Delta y)}_{\text{smaller}}$$

If  $\nabla f(x_0, y_0) \neq (0, 0)$

then  $\Delta z \geq 0$  if  $(\Delta x, \Delta y) \parallel \nabla f / |\nabla f|$   
 $\Delta z < 0$  if  $(\Delta x, \Delta y) \parallel -\nabla f / |\nabla f|$

Def.  $(x_0, y_0)$  is an (interior) critical point of  $f$   
if  $\nabla f(x_0, y_0) = (0, 0)$  or does not exist

It can be shown (P883,  $n=2$ ) that

$$\Delta f = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y) \quad \dots \quad \Delta_1$$

$$+ \frac{1}{2} \left( f_{xx}(x_0, y_0) (\Delta x)^2 + 2f_{xy}(x_0, y_0) \Delta x \Delta y + f_{yy}(x_0, y_0) (\Delta y)^2 \right) \dots \Delta_2$$

P464

$$+ O\left((\Delta x)^3, (\Delta x)^2 \Delta y, \Delta x (\Delta y)^2, (\Delta y)^3\right) \dots \Delta_3$$

$\Delta_{1,2,3}$  = (linear, quadratic, cubic) in  $\Delta x, \Delta y$

$$|\Delta_3| \leq C_1 (|\Delta x|^3 + |\Delta x|^2 |\Delta y| + |\Delta x| |\Delta y|^2 + |\Delta y|^3)$$

If  $\nabla f(x_0, y_0) = 0$ ,  $\Delta_1 = 0$

$$\Delta_2 = A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2 = \text{leading term}$$

$$A = f_{xx}(x_0, y_0), \quad B = f_{xy}(x_0, y_0), \quad C = f_{yy}(x_0, y_0)$$

$$\Delta_3 \ll \Delta_2 \quad \left( \lim_{\Delta_2} \frac{\Delta_3}{\Delta_2} = 0 \right) \quad \text{if } \Delta_2 \neq 0$$

leading term =  $\Delta_2$

$$= A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2 \dots (*)$$

$$= A\left(\left(\Delta x + \frac{B}{A}\Delta y\right)^2 - \frac{B^2 - AC}{A^2}(\Delta y)^2\right)$$

=  $\begin{cases} A \cdot (\text{sum of squares}) & \text{if } B^2 - AC < 0 \\ A \cdot (\text{difference of squares}) & \text{if } B^2 - AC > 0 \end{cases}$

If  $(\Delta x, \Delta y) \neq (0, 0)$

$$\begin{cases} \Delta_2 \text{ always } > 0 & \Leftrightarrow \begin{cases} A > 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ always } < 0 & \Leftrightarrow \begin{cases} A < 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ can change sign} & \Leftrightarrow \begin{cases} B^2 - AC < 0 \\ B^2 - AC > 0 \end{cases} \end{cases}$$

The Second Derivative Test:  $(f_x = 0, f_y = 0)$

$$\begin{cases} f_{xx} > 0, & f_{xy}^2 - f_{xx}f_{yy} < 0: \text{local min} \\ f_{xx} < 0, & f_{xy}^2 - f_{xx}f_{yy} < 0: \text{local max} \\ & f_{xy}^2 - f_{xx}f_{yy} > 0: \text{Saddle point} \end{cases}$$

If  $B^2 - AC > 0$

$$\Delta_2 = A \cdot \left( \Delta x - \frac{-B + \sqrt{B^2 - AC}}{A} \Delta y \right) \cdot \left( \Delta x - \frac{-B - \sqrt{B^2 - AC}}{A} \Delta y \right)$$

$\Delta_2 < 0$

$\Delta_2 > 0$

$(x_0, y_0)$

$\Delta_2 > 0$

$\Delta_2 < 0$



Def:  $(x_0, y_0)$  is a saddle point

of  $f$

if  $\begin{cases} \nabla f = (0, 0) \\ f_{xy} - f_{xx}f_{yy} > 0 \end{cases}$

at  $(x_0, y_0)$

Summary: How to find interior local extremes of  $f$ , assuming all partial derivatives of  $f$  are cont.  $\rightarrow$

Step 1: Find all critical points  $(x_0, y_0)$  where  $\nabla f(x_0, y_0) = (0, 0)$

Step 2: Evaluate  $D = (f_{xy}^2 - f_{xx}f_{yy})(x_0, y_0)$

and  $A = f_{xx}(x_0, y_0)$

(i) If  $A < 0$ ,  $D < 0 \Rightarrow$  local max

(ii) If  $A > 0$ ,  $D < 0 \Rightarrow$  local min

(iii) If  $D > 0 \Rightarrow$  Saddle point (not local max)  
(not local min)

(iv) If  $D = 0 \Rightarrow$  inconclusive

Eg1. find local extremes  
of  $f(x, y) = x^2 + y^2 - 4y + 9$

Sol. Method 1

$$f_x = 2x, \quad f_y = 2(y - 2)$$

$\Rightarrow$  critical point =  $(0, 2)$  only

$$f_{xx}(0, 2) = 2, \quad f_{xy}(0, 2) = 0$$

$$f_{yy}(0, 2) = 2 \Rightarrow D = -4$$

$$A > 0, \quad D < 0$$

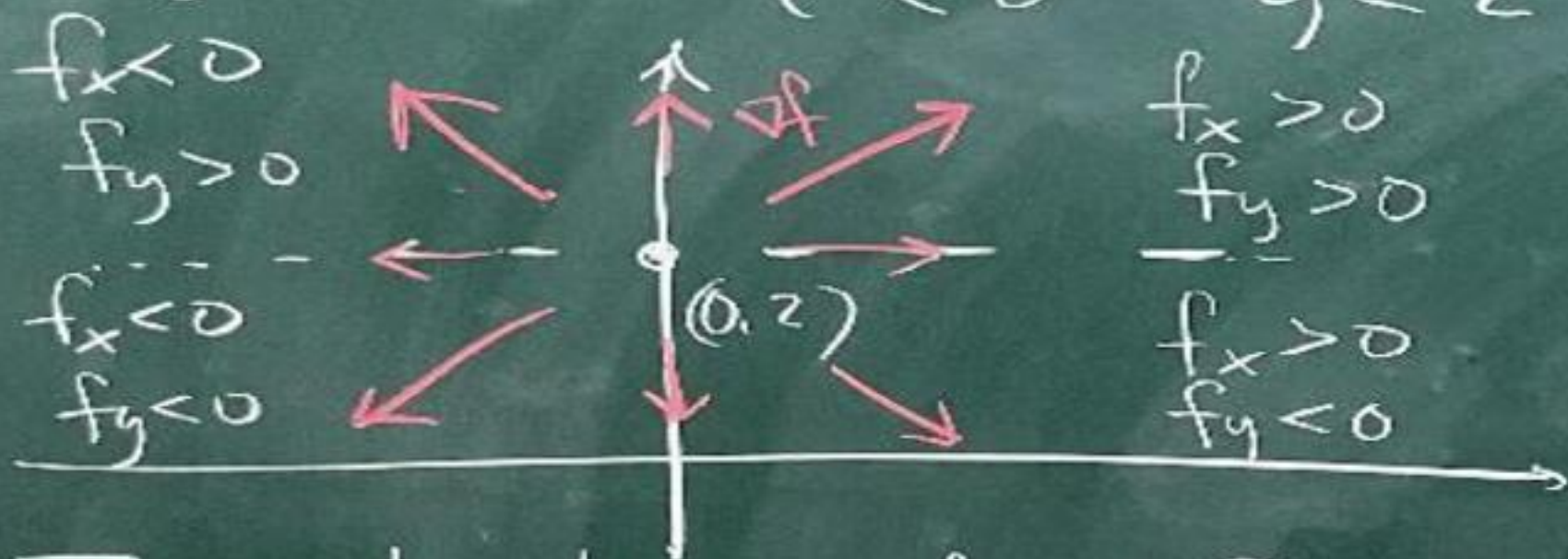
$$\Rightarrow \text{local min} = f(0, 2) = 5$$

local max: none

## Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

$$f_y = 2(y-2) \begin{cases} > 0 & \text{if } y > 2 \\ < 0 & \text{if } y < 2 \end{cases}$$



From directions of  $\nabla f$  near  $(x_0, y_0) = (0, 2)$

$\Rightarrow f(0, 2)$  is a local min

Remark: Near a critical point  
where  $\nabla f(x_0, y_0) = (0, 0)$



$\nabla f$  points outward  
 $\Rightarrow$  local min



$\nabla f$  points inward  
 $\Rightarrow$  local max



$\nabla f$  points inward  
in some directions  
and outward in some  
directions  $\Rightarrow$  Not local min  
Not local max  
(eg. saddle points)