

Eg 2. Find the tangent plane and normal line of $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$ at $(1, 2, 3)$.

Sol tangent plane $\perp \nabla F(1, 2, 3)$
normal line $\parallel \nabla F(1, 2, 3)$

$$F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$$

$$\text{Surface: } F(x, y, z) = 3$$

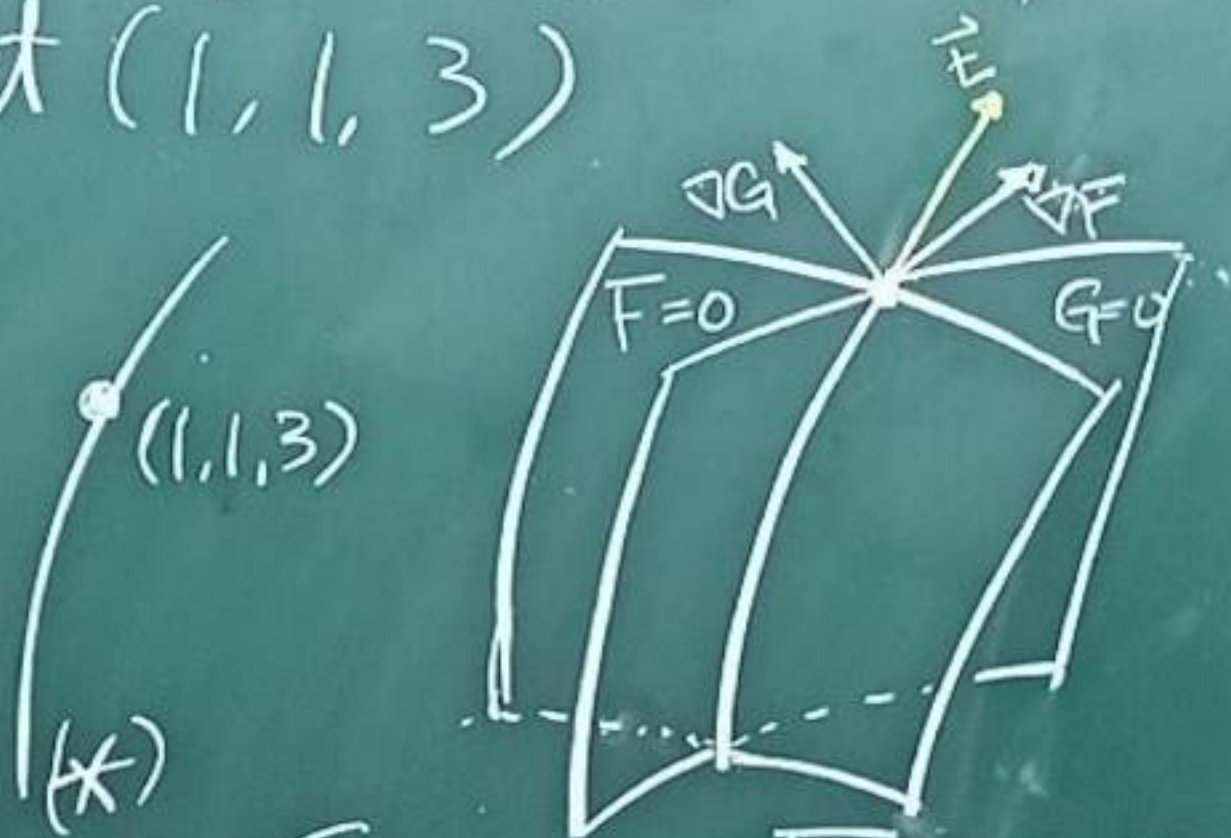
$$\nabla F = (F_x, F_y, F_z) = \left(2x, \frac{y}{2}, \frac{2z}{9}\right)$$

$$\nabla F(1, 2, 3) = \left(2, 1, \frac{2}{3}\right)$$

$$\text{tangent plane: } (x-1, y-2, z-3) \cdot \left(2, 1, \frac{2}{3}\right) = 0$$

$$\text{normal line } \begin{cases} x(t) = 1 + 2t \\ y(t) = 2 + t \\ z(t) = 3 + \frac{2}{3}t \end{cases}$$

Eg3 Find the tangent
 line of $\begin{cases} x^2 + y^2 - 2 = 0 \\ x + z = 4 \end{cases} (*)$
 at $(1, 1, 3)$



If $C = \left\{ (x, y, z), \begin{array}{l} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{array} \right\}$
 and $(x_0, y_0, z_0) \in C$

\vec{t} : tangent vector of C
 at (x_0, y_0, z_0) . Here $F(x, y, z) = x^2 + y^2 - 2$
 $G(x, y, z) = x + z - 4$

$$C \subset S_1 =$$

Since $C \subset S_1 = \left\{ (x, y, z), F(x, y, z) = \text{const} \right\}$
 $= F(x_0, y_0, z_0)$

$\Rightarrow \vec{T}$ is a tangent vector of S_1
at (x_0, y_0, z_0)

$$\Rightarrow \vec{T} \perp \nabla F(x_0, y_0, z_0)$$

Similarly, $\vec{T} \perp \nabla G(x_0, y_0, z_0)$

$$\therefore \vec{T} \parallel \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$$

$$= (2x_0, 2y_0, 0) \times (1, 0, 1)$$

$$= (2, 2, 0) \times (1, 0, 1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2, -2, -2)$$

$$\text{Ans } \frac{x-1}{2} = \frac{y-2}{-2} = \frac{z-3}{-2}$$

$$\text{or } x(t) = 1 + 2t, \quad y(t) = 2 - 2t, \quad z(t) = 3 - 2t$$

Linearization

If $f(x, y)$ is diff. at (x_0, y_0)

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$\therefore L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
is a good approximation of $z = f(x, y)$
near $(x_0, y_0, f(x_0, y_0))$

Linearization: use $L(x, y)$ to approx. $f(x, y)$

$$\Leftrightarrow \Delta z \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \quad (\star)$$

($df = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$)

Note: If " \approx " holds in (\star)

$$\Rightarrow z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow z = L(x, y)$$

Ex 1: $f(x, y) = x^2 - xy + \frac{y^2}{2} + 3$

Find approx. value of $f(3.1, 1.9)$

Sol: take $(x_0, y_0) = (3, 2)$

$$f(3, 2) = 8, \quad f_x(3, 2) = 4$$

$$f_y(3, 2) = -1$$

App. value = $L(3.1, 1.9)$

$$= f(3, 2) + f_x(3, 2) \cdot (3.1 - 3) + f_y(3, 2) \cdot (1.9 - 2)$$

$$= 8 + 4 \cdot 0.1 + (-1) \cdot (-0.1) = 8.5$$

Remark: The same as

$$\Delta z = z - f(3, 2) \approx f_x(3, 2) \Delta x + f_y(3, 2) \Delta y$$

$$\Rightarrow z \approx 8.5$$

Error of linearization

$$f(x, y) \approx L(x, y)$$

How large is $|f(x, y) - L(x, y)|$?

1D case ($y = f(x)$)

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{L(x)} + \underbrace{\frac{f''(\xi)}{2}(x - x_0)^2}_{\text{error}}$$

(Taylor's formula ($n=1$) on P 883)

2D case ($z = f(x, y)$)

$$f(x, y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}$$

$$+ \frac{1}{2} \left(f_{xx}(\xi, \eta)(x - x_0)^2 + 2f_{xy}(\xi, \eta)(x - x_0)(y - y_0) + f_{yy}(\xi, \eta)(y - y_0)^2 \right)$$

where (ξ, η) lies on the segment $(x_0, y_0) - (x, y)$ error

Corollary: If $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all continuous in an open region R , $(x_0, y_0) \in R$

and $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ on R

Then $|f(x, y) - L(x, y)|$

$$\leq \frac{1}{2} (M|x-x_0|^2 + 2M|x-x_0||y-y_0| + M|y-y_0|^2)$$

$$= \frac{M}{2} (|x-x_0| + |y-y_0|)^2$$

Error estimate for linear approximation

In Eg 1, $f_{xx} = 2, f_{xy} = -1, f_{yy} = 1$ ($\therefore M = 2$)

$$\therefore |f(3.1, 1.9) - 8.5| \leq \frac{M}{2} (0.1 + 0.1)^2 = 0.04$$

Eg 2: Volume of Cylinder

$$V(r, h) = \pi r^2 h$$

If $r_0 = 1, h_0 = 5$

$$r_1 = r_0 + \Delta r, \quad \Delta r = 0.1$$

$$h_1 = h_0 + \Delta h, \quad \Delta h = -0.1$$

$$\left[\begin{array}{l} \therefore \\ |\text{Error}| \\ \leq \frac{10\pi}{2} (0.1 + 0.1)^2 \\ = 0.2\pi \end{array} \right]$$

$$\Delta V = V(r_1, h_1) - V(r_0, h_0) = ?$$

Ans: $\Delta V \approx V_r(r_0, h_0) \Delta r + V_h(r_0, h_0) \Delta h$

$$= 2\pi r_0 h_0 \Delta r + \pi r_0^2 \Delta h = 0.9\pi$$

$$|\text{Error}| \leq \frac{M}{2} (|\Delta r| + |\Delta h|)^2$$

Find M , such that $|V_{rr}|, |V_{rh}|, |V_{hh}| \leq M$

$$V_{rr}(C_1, C_2) = 2\pi C_2 \quad (r_0, h_0) = (1, 5)$$

$$V_{rh}(C_1, C_2) = 2\pi C_1$$

$$|C_1| \leq 1.1, |C_2| \leq 5 \Rightarrow M = 10\pi \text{ will do}$$



Extreme Values and Saddle points

How to find local min/max of $f(x, y)$
(assuming all partial derivatives are cont.)

(1) First derivative test

f has local min/max at (x_0, y_0)

$$\Leftrightarrow \nabla f(x_0, y_0) = (0, 0) \quad \text{leading term}$$

$$\therefore \Delta z = f(x, y) - f(x_0, y_0) = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y) + \underbrace{(\varepsilon_1, \varepsilon_2) \cdot (\Delta x, \Delta y)}_{\text{smaller}}$$

If $\nabla f(x_0, y_0) \neq (0, 0)$

then $\Delta z > 0$ if $(\Delta x, \Delta y) \parallel \nabla f / |\nabla f|$
 $\Delta z < 0$ if $(\Delta x, \Delta y) \parallel (-\nabla f / |\nabla f|)$

Def. (x_0, y_0) is an (interior) critical point of f
if $\nabla f(x_0, y_0) = (0, 0)$ or does not exist

It can be shown (P883, n=2) that

$$\Delta f = \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y) \quad \dots \quad \Delta_1$$

$$+ \frac{1}{2} \left(f_{xx}(x_0, y_0) (\Delta x)^2 + 2f_{xy}(x_0, y_0) \Delta x \Delta y + f_{yy}(x_0, y_0) (\Delta y)^2 \right) \dots \Delta_2$$

P464

$$+ O((\Delta x)^3, (\Delta x)^2 \Delta y, \Delta x (\Delta y)^2, (\Delta y)^3) \dots \Delta_3$$

$\Delta_{1,2,3}$ = (linear, quadratic, cubic) in $\Delta x, \Delta y$

$$|\Delta_3| \leq C_1 (|\Delta x|^3 + |\Delta x|^2 |\Delta y| + |\Delta x| |\Delta y|^2 + |\Delta y|^3)$$

If $\nabla f(x_0, y_0) = 0$, $\Delta_1 = 0$

$$\Delta_2 = A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2 = \text{leading term}$$

$$A = f_{xx}(x_0, y_0), \quad B = f_{xy}(x_0, y_0), \quad C = f_{yy}(x_0, y_0)$$

$$\Delta_3 \ll \Delta_2 \quad \left(\lim_{\Delta_2} \frac{\Delta_3}{\Delta_2} = 0 \right) \quad \text{if } \Delta_2 \neq 0$$

leading term = Δ_2

$$= A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2 \dots (*)$$

$$= A\left(\left(\Delta x + \frac{B}{A}\Delta y\right)^2 - \frac{B^2 - AC}{A^2}(\Delta y)^2\right)$$

= $\begin{cases} A \cdot (\text{sum of squares}) & \text{if } B^2 - AC < 0 \end{cases}$

$\begin{cases} A \cdot (\text{difference of squares}) & \text{if } B^2 - AC > 0 \end{cases}$

If $(\Delta x, \Delta y) \neq (0, 0)$

$$\begin{cases} \Delta_2 \text{ always } > 0 \iff \begin{cases} A > 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ always } < 0 \iff \begin{cases} A < 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ can change sign} \iff \begin{cases} B^2 - AC < 0 \\ B^2 - AC > 0 \end{cases} \end{cases}$$

The Second Derivative Test: $(f_x = 0, f_y = 0)$

$$\begin{cases} f_{xx} > 0, \quad f_{xy}^2 - f_{xx}f_{yy} < 0: \text{ local min} \\ f_{xx} < 0, \quad f_{xy}^2 - f_{xx}f_{yy} < 0: \text{ local max} \\ f_{xy}^2 - f_{xx}f_{yy} > 0: \text{ Saddle point} \end{cases}$$

If $B^2 - AC > 0$

$$\Delta_2 = A \cdot \left(\Delta x - \frac{-B + \sqrt{B^2 - AC}}{A} \Delta y \right) \cdot \left(\Delta x - \frac{-B - \sqrt{B^2 - AC}}{A} \Delta y \right)$$



Def: (x_0, y_0) is a saddle point

of f
if $\begin{cases} \nabla f = (0, 0) \\ f_{xy} - f_{xx}f_{yy} > 0 \end{cases}$ at (x_0, y_0)

Summary: How to find interior local extremes of f , assuming all partial derivatives of f are cont.?

Step 1. Find all critical points (x_0, y_0) where $\nabla f(x_0, y_0) = (0, 0)$

Step 2. Evaluate $D = (f_{xy}^2 - f_{xx}f_{yy})(x_0, y_0)$ and $A = f_{xx}(x_0, y_0)$

(i) If $A < 0, D < 0 \Rightarrow$ local max

(ii) If $A > 0, D < 0 \Rightarrow$ local min

(iii) If $D > 0 \Rightarrow$ Saddle point (not local max)
(not local min)

(iv) If $D = 0 \Rightarrow$ inconclusive

Ex 1. find local extremes

$$\text{of } f(x, y) = x^2 + y^2 - 4y + 9$$

Sol. Method 1

$$f_x = 2x, \quad f_y = 2(y - 2)$$

\Rightarrow critical point = $(0, 2)$ only

$$f_{xx}(0, 2) = 2, \quad f_{xy}(0, 2) = 0$$

$$f_{yy}(0, 2) = 2 \Rightarrow D = -4$$

$$A > 0, \quad D < 0$$

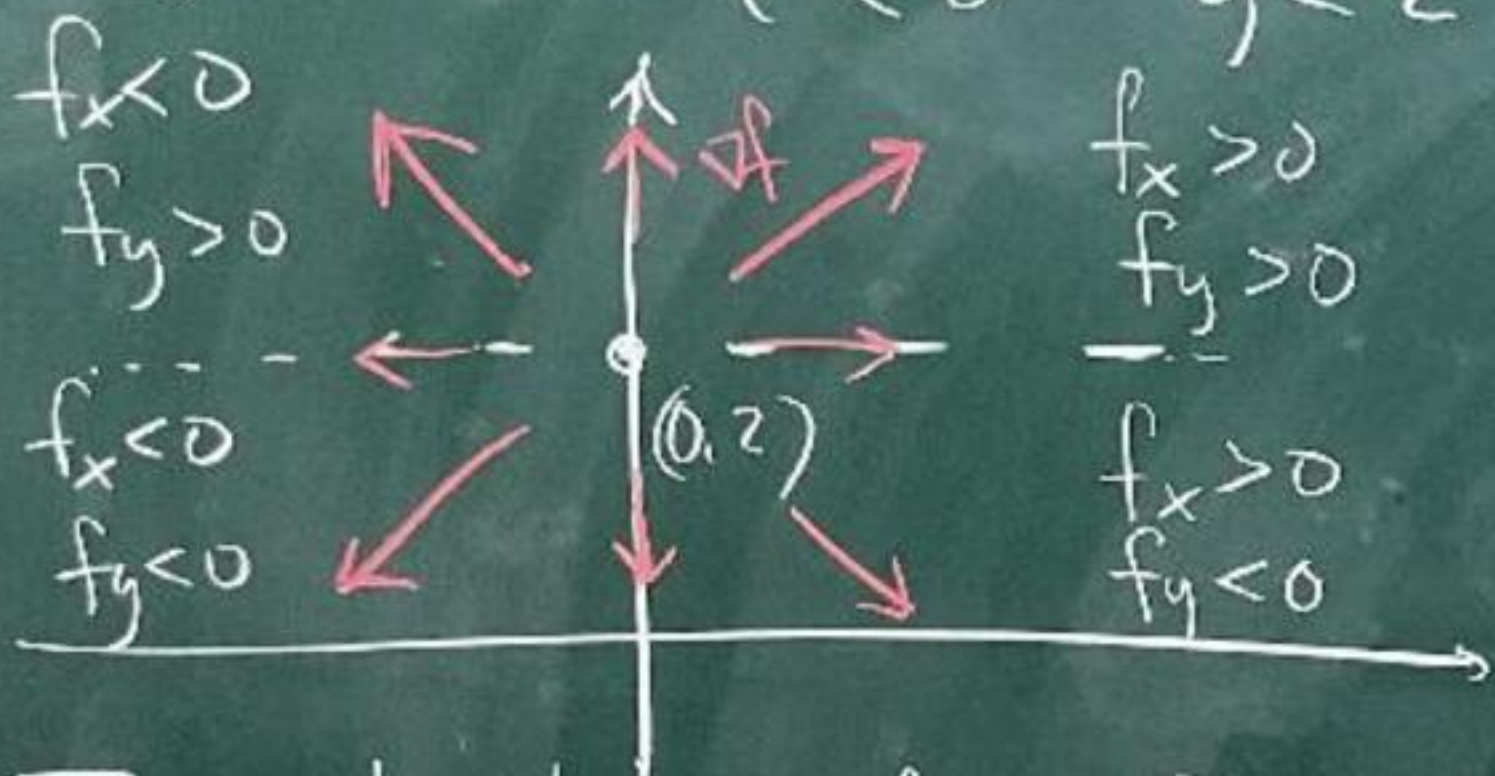
$$\Rightarrow \text{local min} = f(0, 2) = 5$$

local max: none

Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} > 0 & \text{if } x > 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

$$f_y = 2(y-2) \begin{cases} > 0 & \text{if } y > 2 \\ < 0 & \text{if } y < 2 \end{cases}$$



From directions of ∇f near
 $(x_0, y_0) = (0, 2)$

$\Rightarrow f(0, 2)$ is a local min

Remark: Near a critical point where $\nabla f(x_0, y_0) = (0, 0)$



∇f points outward
 \Rightarrow local min



∇f points inward
 \Rightarrow local max



∇f points inward in some directions and outward in some directions \Rightarrow Not local min
Not local max
(eg. saddle points)