Eg2. Find the tangent plane and normal line of $x^2+\frac{y^2}{4}+\frac{z^2}{9}=3$ at $(1,2,3)$.

Sol tangent plane $\perp \nabla F(1,2,3)$

normal line $\parallel \nabla F(1,2,3)$

$F(x,y,z)=x^2+\frac{y^2}{4}+\frac{z^2}{9}$

Surface: $F(x,y,z)=3$

$\nabla F=(F_x,F_y,F_z)=(2x,\frac{y}{2},\frac{2z}{9})$

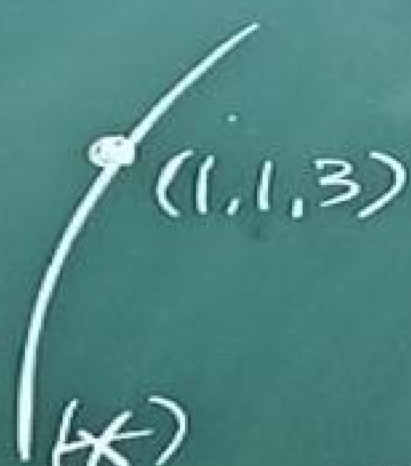
$\nabla F(1,2,3)=(2,1,\frac{2}{3})$

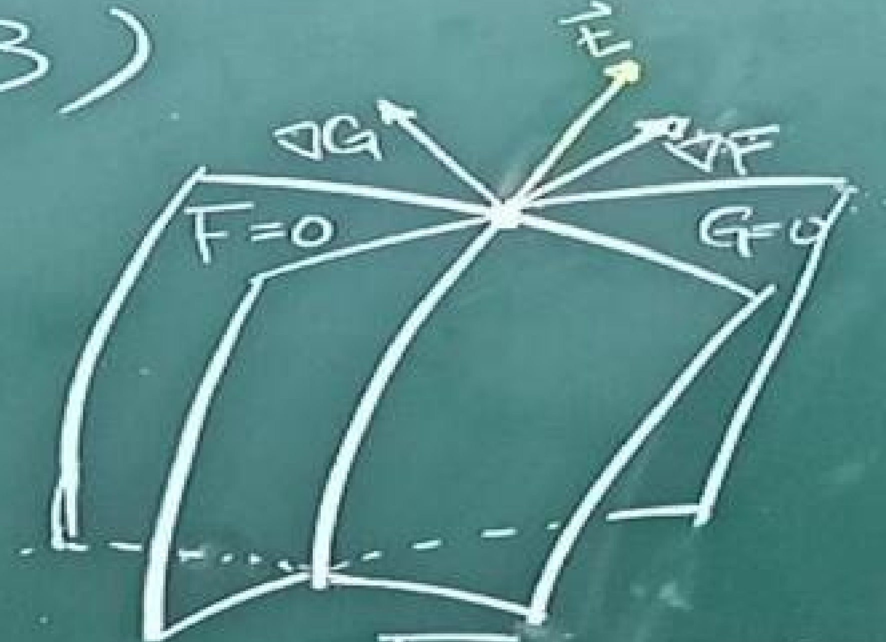
tangent plane: $(x-1,y-2,z-3)\cdot(2,1,\frac{2}{3})=0$

normal line $x(t)=1+2t$

$y(t)=2+t$, $z(t)=3+\frac{2}{3}t$

Eg3 Find the tangent line of $\begin{cases} x^2+y^2-2=0 \\ x+z=4 \end{cases}$ (*) at $(1,1,3)$





If $C = \left\{ (x,y,z),\ \begin{matrix} F(x,y,z)=0 \\ G(x,y,z)=0 \end{matrix} \right\}$

and $(x_0, y_0, z_0) \in C$

$\vec{t}$: tangent vector of C at (x_0, y_0, z_0). Here $F(x,y,z) = x^2+y^2-2$

$G(x,y,z) = x+z-4$

$C \subset S_1 =$

Since $C \subset S_1 = \{(x, y, z) \mid F(x, y, z) = \text{const} = F(x_0, y_0, z_0)\}$

$\Rightarrow \vec{t}$ is a tangent vector of S_1 at (x_0, y_0, z_0)

$\Rightarrow \vec{t} \perp \nabla F(x_0, y_0, z_0)$,

Similarly, $\vec{t} \perp \nabla G(x_0, y_0, z_0)$

$$\therefore \vec{t} \parallel \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$$

$$= (2x_0, 2y_0, 0) \times (1, 0, 1)$$

$$= (2, 2, 0) \times (1, 0, 1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2, -2, -2)$$

Ans $\dfrac{x-1}{2} = \dfrac{y-2}{-2} = \dfrac{z-3}{-2}$

or $x(t) = 1 + 2t,\ y(t) = 2 - 2t,\ z(t) = 3 - 2t$

Linearization

If $f(x,y)$ is diff. at (x_0, y_0)

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

$\therefore L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

is a good approximation of $z = f(x,y)$

near $(x_0, y_0, f(x_0,y_0))$

Linearization: use $L(x,y)$ to approx. $f(x,y)$

$$\Leftrightarrow \quad \Delta z \cong f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y \quad (\star)$$

$$(\, df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy \,)$$

Note: If "=" holds in $(\star)$

$$\Rightarrow z - \overset{=f(x_0,y_0)}{z_0} = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$\Rightarrow z = L(x,y)$$

Eg 1: $f(x,y) = x^2 - xy + \frac{y^2}{2} + 3$

Find approx. value of $f(3.1, 1.9)$

<u>Sol</u>: take $(x_0, y_0) = (3,2)$

$f(3,2) = 8$, $f_x(3,2) = 4$

$f_y(3,2) = -1$

App. value $= L(3.1, 1.9)$

$= f(3,2) + f_x(3,2)\cdot(3.1-3) + f_y(3,2)(1.9-2)$

$= 8 + 4\cdot 0.1 + (-1)\cdot(-0.1) = 8.5$

<u>Remark</u>: the same as

$\Delta z = z - f(3,2) \cong f_x(3,2)\Delta x + f_y(3,2)\Delta y$

$\Rightarrow z \cong 8.5$

Error of linearization

$f(x,y) \cong L(x,y)$

How large is $|f(x,y) - L(x,y)|$?

1D case ($y = f(x)$)

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{L(x)} + \underbrace{\frac{f''(c)}{2}(x-x_0)^2}_{\text{error}}$$

(Taylor's formula ($n=1$) on P 883)

2D case ($z = f(x,y)$)

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{L(x,y)}$$
$$\underbrace{+ \frac{1}{2}\left(f_{xx}(c_1,c_2)(x-x_0)^2 + 2f_{xy}(c_1,c_2)(x-x_0)(y-y_0) + f_{yy}(c_1,c_2)(y-y_0)^2\right)}_{\text{error}}$$

where (c_1,c_2) lies on the segment (x_0,y_0) (x,y)

Corollary: If $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all continuous in an open region R, $(x_0, y_0) \in R$ and $|f_{xx}|, |f_{xy}|, |f_{yy}| \le M$ on R

Then $|f(x,y) - L(x,y)|$

$$\le \frac{1}{2}\left(M|x-x_0|^2 + 2M|x-x_0||y-y_0| + M|y-y_0|^2\right)$$

$$= \underbrace{\frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2}_{\text{Error estimate for linear approximation}}$$

In Eg 1, $f_{xx} = 2$, $f_{xy} = -1$, $f_{yy} = 1$ ($\therefore M = 2$)

$\therefore |f(3.1, 1.9) - 8.5| \le \frac{M}{2}(0.1 + 0.1)^2 = 0.04$

Eg2: Volume of Cylinder

$V(r,h) = \pi r^2 h$

If $r_0 = 1$, $h_0 = 5$

$r_1 = r_0 + \Delta r$, $\Delta r = 0.1$

$h_1 = h_0 + \Delta h$, $\Delta h = -0.1$

$\Delta V = V(r_1, h_1) - V(r_0, h_0) = ?$

$$\left[\ \therefore\ |\text{Error}| \le \frac{10\pi}{2}(0.1+0.1)^2 = 0.2\pi\ \right]$$

<u>Ans</u>: $\Delta V \cong V_r(r_0,h_0)\Delta r + V_h(r_0,h_0)\Delta h$

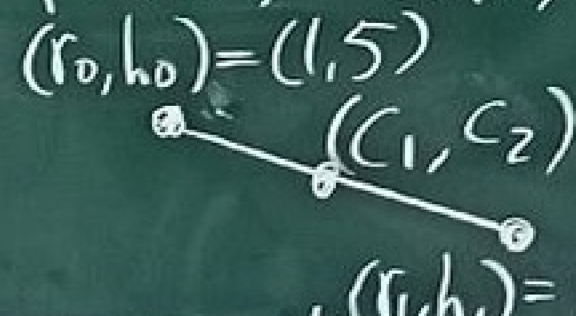
$= 2\pi r_0 h_0 \Delta r + \pi r_0^2 \Delta h = 0.9\pi$

$|\text{Error}| \le \frac{M}{2}\left(|\Delta r| + |\Delta h|\right)^2$

Find M, such that $|V_{rr}|, |V_{rh}|, |\underset{=0}{V_{hh}}| \le M$

$V_{rr}(c_1, c_2) = 2\pi c_2$

$V_{rh}(c_1, c_2) = 2\pi c_1$



$|c_1| \le 1.1$, $|c_2| \le 5 \Rightarrow M = 10\pi$ will do

Extreme Values and Saddle points

How to find local min/max of $f(x,y)$
(assuming all partial derivatives are cont.)

(1) First derivative test

f has local min/max at (x_0, y_0)

$\overset{\Rightarrow}{\not\Leftarrow}$ $\nabla f(x_0, y_0) = (0,0)$

$\because \Delta z = f(x,y) - f(x_0,y_0) = \overbrace{\nabla f(x_0,y_0)\cdot(\Delta x, \Delta y)}^{\text{leading term}}$
$+ \underbrace{(\varepsilon_1, \varepsilon_2)\cdot(\Delta x, \Delta y)}_{\text{smaller}}$

If $\nabla f(x_0, y_0) \neq (0,0)$

then $\Delta z \underset{<0}{>0}$ if $(\Delta x, \Delta y) \parallel \underset{(-\nabla f/|\nabla f|)}{\nabla f/|\nabla f|}$

<u>Def</u>: (x_0, y_0) is an (interior) critical point of f
if $\nabla f(x_0, y_0) = (0,0)$ or does not exist

It can be shown (P883, n=2) that

$$\Delta f = \nabla f(x_0, y_0)\cdot(\Delta x, \Delta y) \quad \cdots \quad \Delta_1$$

$$+\frac{1}{2}\begin{pmatrix} f_{xx}(x_0,y_0)(\Delta x)^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y \\ + f_{yy}(x_0,y_0)(\Delta y)^2 \end{pmatrix} \cdots \quad \Delta_2$$

P464

$$+O\left((\Delta x)^3, (\Delta x)^2\Delta y, \Delta x(\Delta y)^2, (\Delta y)^3\right) \cdots \quad \Delta_3$$

$\Delta_{1,2,3}$ = (linear, quadratic, cubic) in Δx, Δy

$$|\Delta_3| \leq C_1\left(|\Delta x|^3 + |\Delta x|^2|\Delta y| + |\Delta x||\Delta y|^2 + |\Delta y|^3\right)$$

If $\nabla f(x_0, y_0) = 0$, $\Delta_1 = 0$

$\Delta_2 = A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2$ = leading term

$A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, $C = f_{yy}(x_0,y_0)$

$\Delta_3 \ll \Delta_2$ $\left(\lim \frac{\Delta_3}{\Delta_2} = 0\right)$ if $\Delta_2 \neq 0$

leading term $= \Delta_2$

$$= A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2 \quad \cdots (*)$$

$$= A\left(\left(\Delta x + \frac{B}{A}\Delta y\right)^2 - \frac{B^2 - AC}{A^2}(\Delta y)^2\right)$$

$$= \begin{cases} A \cdot (\text{sum of squares}) & \text{if } B^2 - AC < 0 \\ A \cdot (\text{difference of squares}) & \text{if } B^2 - AC > 0 \end{cases}$$

If $(\Delta x, \Delta y) \neq (0,0)$

$$\begin{cases} \Delta_2 \text{ always } > 0 \iff \begin{cases} A > 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ always } < 0 \iff \begin{cases} A < 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ can change sign} \iff B^2 - AC > 0 \end{cases}$$

The Second Derivative Test: $(f_x = 0,\ f_y = 0)$

$$\begin{cases} f_{xx} > 0,\ f_{xy}^2 - f_{xx}f_{yy} < 0 : \text{local min} \\ f_{xx} < 0,\ f_{xy}^2 - f_{xx}f_{yy} < 0 : \text{local max} \\ \phantom{f_{xx} < 0,\ } f_{xy}^2 - f_{xx}f_{yy} > 0 : \text{Saddle point} \end{cases}$$

If $B^2 - AC > 0$

$$\Delta_2 = A \cdot \left(\Delta x - \frac{-B + \sqrt{B^2 - AC}}{A}\Delta y\right) \cdot \left(\Delta x - \frac{-B - \sqrt{B^2 - AC}}{A}\Delta y\right)$$

Def: (x_0, y_0) is a saddle point of f

if $\begin{cases} \nabla f = (0,0) \\ f_{xy}^2 - f_{xx} f_{yy} > 0 \end{cases}$ at (x_0, y_0)

Summary: How to find interior local extremes of f assuming all partial derivatives of f are cont.?

Step 1: Find all critical points (x_0, y_0) where $\nabla f(x_0, y_0)$ where $\nabla f(x_0, y_0) = (0,0)$

Step 2: Evalute $D = (f_{xy}^2 - f_{xx} f_{yy})(x_0, y_0)$ and $A = f_{xx}(x_0, y_0)$

(i) If $A < 0$, $D < 0 \Rightarrow$ local max

(ii) If $A > 0$, $D < 0 \Rightarrow$ local min

(iii) If $D > 0 \Rightarrow$ Saddle point $\left(\begin{matrix}\text{not local max} \\ \text{not local min}\end{matrix}\right)$

(iv) If $D = 0 \Rightarrow$ inconclusive