

Chain Rule

Single variable function

$$\frac{d}{dx} f(g(x)) = \left. \frac{df}{dy} \right|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables function $z = f(x, y)$

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{\substack{t \rightarrow t_0 \\ t=t_0}} \frac{\Delta z}{\Delta t}$$

$$\left(\Delta z = f(x(t), y(t)) - f(x(t_0), y(t_0)) \right)$$

(assume f is differentiable)

$$= \lim_{\substack{\Delta t \rightarrow 0 \\ t \rightarrow t_0}} \left(f_x(x(t), y(t)) + \varepsilon_1 \right) \frac{\Delta x}{\Delta t} + \left(f_y(x(t), y(t)) + \varepsilon_2 \right) \frac{\Delta y}{\Delta t}$$

(*)

When $\Delta t \rightarrow 0$ ($t \rightarrow t_0$)

we have $(x(t), y(t)) \Rightarrow (x(t_0), y(t_0))$

(assume $x(t), y(t)$ are also
differentiable at t_0 ,
therefore continuous at t_0)

$$\Rightarrow (\Delta x, \Delta y) \rightarrow (0, 0)$$

$$\Rightarrow \lim_{\Delta t \rightarrow 0} \varepsilon_1 = 0 = \lim_{\Delta t \rightarrow 0} \varepsilon_2$$

$\therefore (*)$

$$\begin{aligned} &= f_x(x(t_0), y(t_0)) \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \\ &+ f_y(x(t_0), y(t_0)) \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \\ &= f_x(x(t_0), y(t_0)) x'(t_0) + f_y(x(t_0), y(t_0)) y'(t_0) \end{aligned}$$

In other words

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

Similarly, $\partial_s f(x(s, t), y(s, t), z(s, t))$

$$= f_x(x(s, t), y(s, t), z(s, t)) \cdot \partial_s x(s, t)$$

$$+ f_y(x(s, t), y(s, t), z(s, t)) \cdot \partial_s y(s, t)$$

$$+ f_z(x(s, t), y(s, t), z(s, t)) \cdot \partial_s z(s, t)$$

$$\partial_t f(x(s, t), y(s, t), z(s, t))$$

$$= f_x(x(s, t), y(s, t), z(s, t)) \cdot \partial_t x(s, t)$$

$$+ f_y(x(s, t), y(s, t), z(s, t)) \cdot \partial_t y(s, t)$$

$$+ f_z(x(s, t), y(s, t), z(s, t)) \cdot \partial_t z(s, t)$$

Implicit differentiation revisited

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x, y, z) = 0$

(i.e. If $z(x, y)$ is implicitly defined
by $F(x, y, z) = 0$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$)

Sol $F(x, y, z(x, y)) = 0$ (as two functions of (x, y))
 $\Rightarrow \frac{\partial}{\partial x} F(x, y, z(x, y)) = 0$

$$\Rightarrow \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x}(x, y, z(x, y)) + \frac{\partial F}{\partial z}(x, y, z(x, y)) \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} \text{ at } (x, y, z) = - \frac{\frac{\partial F}{\partial x}(x, y, z)}{\frac{\partial F}{\partial z}(x, y, z)} = - \frac{\frac{\partial F}{\partial x}(x, y, z)}{\frac{\partial F}{\partial z}(x, y, z)}$$

Similarly

$$\frac{\partial z}{\partial y} \text{ at } (x, y, z) = - \frac{\partial_y F(x, y, z)}{\partial_z F(x, y, z)}$$

Ex 1 Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at (1, 1, 1)

from $F(x, y, z) = xy + z^3x - 2yz = 0$

Ans. $\partial_x F(1, 1, 1) = y + z^3|_{(1, 1, 1)} = 2$

$$\partial_y F(1, 1, 1) = x - 2z|_{(1, 1, 1)} = -1$$

$$\partial_z F(1, 1, 1) = 3z^2x - 2y|_{(1, 1, 1)} = 1$$

$$\therefore \frac{\partial z}{\partial x} \text{ at } (1, 1, 1) = -\frac{2}{1} = -2$$

$$\frac{\partial z}{\partial y} \text{ at } (1, 1, 1) = -\frac{(-1)}{1} = 1$$

Directional derivative

Def: $\left(\frac{df}{ds}\right)_{\vec{u}, P_0} (= D_{\vec{u}} f(x_0, y_0))$
($P_0 = (x_0, y_0)$)

Derivative of $f(x, y)$ at $P_0 = (x_0, y_0)$ in the direction of the unit vector $\vec{u} = (u_1, u_2)$
($u_1^2 + u_2^2 = 1$)

$\stackrel{\text{def}}{=} \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s - 0}$

$\left(= \lim_{s \rightarrow 0} \frac{f(x(s), y(s)) - f(x(0), y(0))}{s - 0} \right)$
where $x(s) = x_0 + su_1$, $y(s) = y_0 + su_2$

Thm If $f(x, y)$ is differentiable
at (x_0, y_0) , then $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$
where $\nabla f = (f_x, f_y)$ (gradient of f)

pf: $\Delta z = (f_x(x_0, y_0) + \varepsilon_1) \Delta x + (f_y(x_0, y_0) + \varepsilon_2) \Delta y$

Let $x(s) = x_0 + s u_1$, $y(s) = y_0 + s u_2$

$$\Delta x = x(s) - x(0) = s u_1 \quad \Delta s = s - 0$$

$$\Delta y = y(s) - y(0) = s u_2$$

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \lim_{\Delta s \rightarrow 0} \frac{\Delta z}{\Delta s}$$

$$= \lim_{\Delta s \rightarrow 0} \left(f_x(x_0, y_0) + 0 \right) \frac{\Delta x}{\Delta s} + \left(f_y(x_0, y_0) + 0 \right) \frac{\Delta y}{\Delta s}$$
$$= f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2 = \nabla f(x_0, y_0) \cdot \vec{u}$$

Eg 2 $f(x, y) = x^2 + xy$

Note: f is differentiable

let $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$D_{\vec{u}} f(1, 2) = ?$$

Sol $= \nabla f(1, 2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$= \frac{f_x(1, 2)}{\sqrt{2}} + \frac{f_y(1, 2)}{\sqrt{2}}$$

$$f_x = 2x + y, \quad f_y = x$$

$$\text{Ans} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$$

$$\text{Ex 3 } f(x,y) = \begin{cases} \frac{4xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Evaluate $D_{\vec{u}} f(0,0)$ and $\nabla f(0,0) \cdot \vec{u}$

Ans: $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x-0} = 0$

$$f_y(0,0) = \lim_{y \rightarrow 0} \frac{f(0,y) - f(0,0)}{y-0} = 0$$

$$D_{\vec{u}} f(0,0) = \lim_{s \rightarrow 0} \frac{f(su_1, su_2) - f(0,0)}{s-0} \quad (\underline{\underline{u_1^2 + u_2^2 = 1}})$$

$$= \lim_{s \rightarrow 0} \frac{\frac{4s^3 u_1 u_2}{s^2}}{s-0} = 4u_1 u_2 \neq \nabla f(0,0) \cdot \vec{u}$$

$\parallel$
 $(0,0)$

Ex 4 $f(x,y) = ax + by + c$

$\nabla f(0,0) \cdot \vec{u} \neq D_{\vec{u}} f(0,0)$

Sol $f_x(0,0) = a, f_y(0,0) = b$

$$D_{\vec{u}} f(0,0) = \lim_{s \rightarrow 0} \frac{(a s u_1 + b s u_2 + c) - c}{s}$$

$$= a u_1 + b u_2 = \nabla f(0,0) \cdot \vec{u}$$

Remark. f is diff. at (x_0, y_0)

$\Rightarrow Z = f(x,y)$ has a tangent plane at $(x_0, y_0, f(x_0, y_0))$

$\Rightarrow f$ is close to a linear function

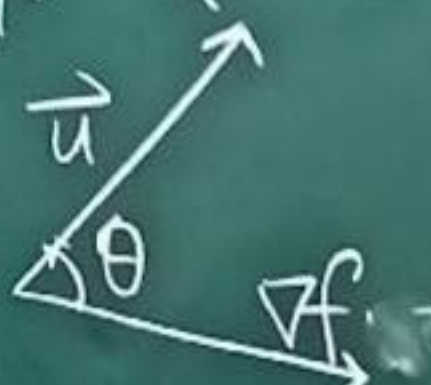
$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

Properties of ∇f

If f is diff. at (x_0, y_0)

fix

for

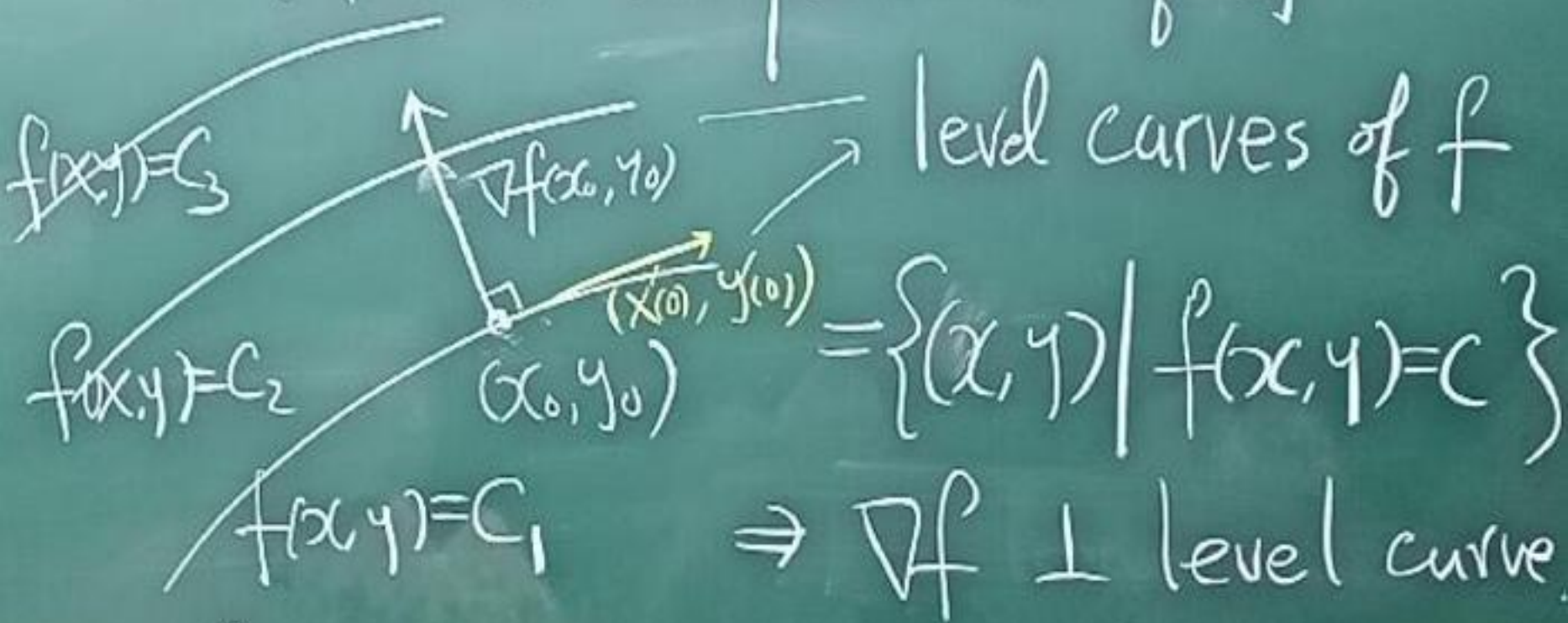
$$\begin{aligned}\Rightarrow D_{\vec{u}} f(x_0, y_0) &= \nabla f(x_0, y_0) \cdot \vec{u} \\ &= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cos \theta\end{aligned}$$


The diagram shows two vectors, $\vec{u}$ and ∇f , originating from the same point. The angle between them is labeled θ . The vector $\vec{u}$ is pointing upwards and to the right, while ∇f is pointing downwards and to the right.

$\Rightarrow$ (1) f increase most rapidly
(decrease) in the direction of $\vec{u}$ if $\cos \theta = 1$
(-1)
i.e. in the direction of ∇f
($-\nabla f$)

$$(2) D_{\vec{u}} f(x_0, y_0) = 0 \Leftrightarrow \nabla f(x_0, y_0) \perp \vec{u}$$

(3) Another interpretation of ∇f



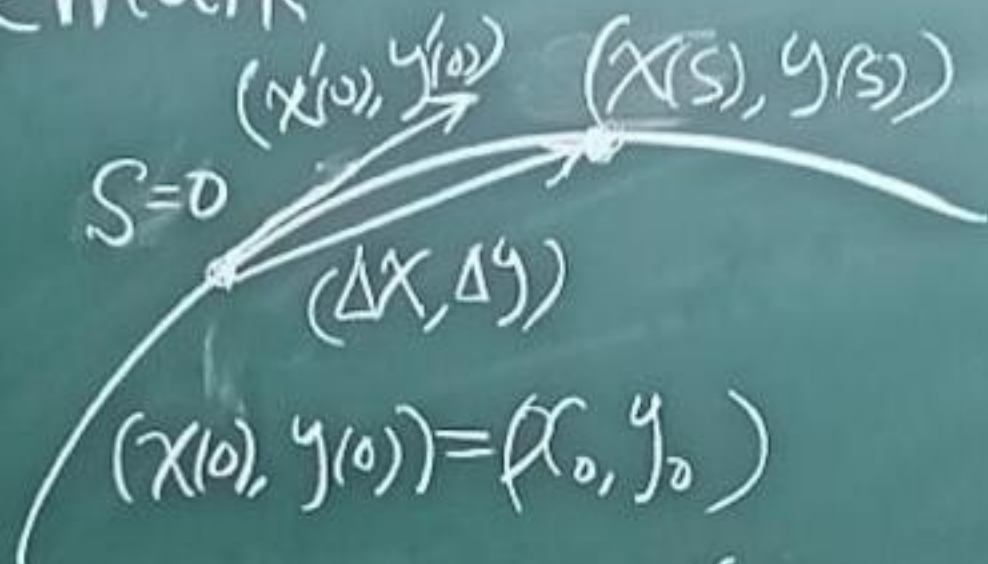
pf: Let $(x(s), y(s))$ be a level curve of f (differentiable) with $(x(0), y(0)) = (x_0, y_0)$

$$\Rightarrow f(x(s), y(s)) = \text{constant}$$

$$\frac{d}{ds} \Big|_{s=0} \Rightarrow \nabla f(x_0, y_0) \cdot (x'(0), y'(0)) = 0$$

$(x'(0), y'(0))$ = a tangent vector of level curve

Remark



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s \rightarrow 0} \begin{pmatrix} \frac{x(s) - x(0)}{s - 0} \\ \frac{y(s) - y(0)}{s - 0} \end{pmatrix}$$

$$= \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad \Delta s = s - 0$$

= a tangent vector

$\therefore (x'(0), y'(0))$ points in the direction of tangent line at $(x(0), y(0))$

Eg1 Find the tangent line
(normal)

of $\frac{x^2}{4} + y^2 = 2$ at $(-2, 1)$

Sol Let $f(x, y) = \frac{x^2}{4} + y^2$

$\therefore \frac{x^2}{4} + y^2 = 2$ is a level curve
of f

$\therefore \nabla f(-2, 1) =$ normal vector of
the level curve at $(-2, 1)$

Tangent line: $(-1, 2)$

$$(x+2, y-1) \cdot \nabla f(-2, 1) = 0$$

Normal line:

$$\frac{y-1}{x-2} = \frac{2}{-1}$$

Algebraic Rule for gradient

$$(i) \nabla(f \pm g) = \nabla f \pm \nabla g$$

$$(ii) \nabla(cf(x,y)) = c \nabla f(x,y)$$

$$(iii) \nabla(f \cdot g) = f \nabla g + g \nabla f$$

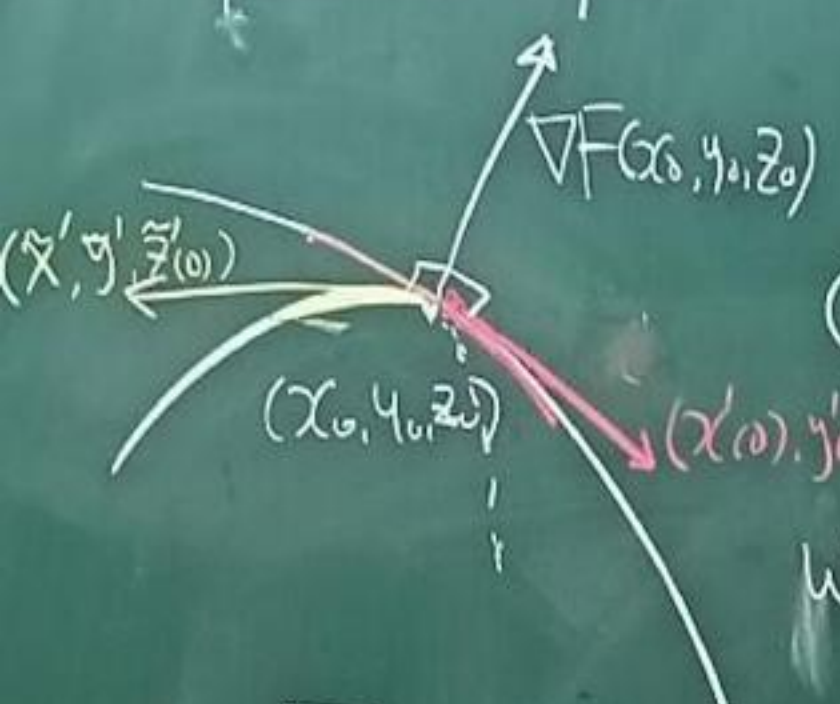
$$(iv) \nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$$

Pf: Check x -component
and y -component directly

Tangent plane and normal line
of level surface of $F(x, y, z)$ (diff.)

$$\{ (x, y, z) \mid F(x, y, z) = C \}$$

Suppose $\{x(t), y(t), z(t)\}$
is a level surface of F
at (x_0, y_0, z_0)
with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$



$$\Rightarrow F(x(t), y(t), z(t)) = \text{constant} = F(x_0, y_0, z_0)$$

$$\frac{d}{dt} \Big|_{t=0} \Rightarrow \nabla F(x_0, y_0, z_0) \cdot \underbrace{(x'(0), y'(0), z'(0))}_{\text{a tangent vector of the level surface at } (x_0, y_0, z_0)} = 0$$

a tangent vector
of the level surface at (x_0, y_0, z_0)

For any curve $(x(t), y(t), z(t))$
on the surface passing (x_0, y_0, z_0)

$$\nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0))$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{matrix} \text{any tangent} \\ \text{vector at } (x_0, y_0, z_0) \end{matrix}$$

$$\Rightarrow \nabla F(x_0, y_0, z_0) \perp \begin{matrix} \text{tangent plane at} \\ (x_0, y_0, z_0) \end{matrix}$$

tangent plane

$$(x - x_0, y - y_0, z - z_0) \cdot \nabla F(x_0, y_0, z_0) = 0$$

$$\text{normal line: } \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\text{or } \begin{cases} x(t) = x_0 + F_x(x_0, y_0, z_0)t \\ y(t) = y_0 + F_y(x_0, y_0, z_0)t \\ z(t) = z_0 + F_z(x_0, y_0, z_0)t \end{cases}$$